Ralph's Task Balance

Ralph is a manager for Alice's firm. Ralph is responsible for balancing his skill, effort, and time across multiple tasks. Alice recognizes task balance is subtle and is concerned about supplying appropriate incentives to help Ralph achieve balance. Alice has identified two measures, x and y, to aid with this task.

$$\begin{aligned} x &= a_1 + \alpha a_2 + \varepsilon_x \\ y &= a_1 + \gamma a_2 + \varepsilon_y \end{aligned}$$

where a_1 and a_2 represent Ralph's input $a \in \{H, L\}$ to task 1 and task 2, respectively, $a_1 + a_2 \leq H$ if H is supplied or $a_1 + a_2 \leq L$ if L is supplied and ε_x and ε_y are independent, mean zero normally distributed noise terms with variance σ_x^2 and σ_y^2, respectively. Alice compensates Ralph based on a flat wage w (since payments are expressed in normalized form this and the piece rates could possibly be negative) plus incentive-providing "piece rates" (rates which are a function of the performance measures)

$$I = w + \beta_1 x + \beta_2 y$$

Ralph is risk averse with negative exponential utility $U(I, a) = -e^{-\rho(I - c(a))}$ where $c(a) \in \{c_H, c_L\}$ refers to Ralph's personal cost of supplying input $a \in \{H, L\}$. Importantly, Ralph is indifferent between the tasks, only Alice cares about task balance. Since the performance measures are normally distributed and payments are linear (affine actually), Ralph's certainty equivalent can be conveniently expressed as[1]

$$CE(a) = E[I \mid a] - \frac{1}{2}\rho Var[I] - c(a)$$

where

$$\begin{aligned} E[I \mid a] &= w + \beta_1 E[x \mid a] + \beta_2 E[y \mid a] \\ &= w + \beta_1 [a_1 + \alpha a_2] + \beta_2 [a_1 + \gamma a_2] \end{aligned}$$

and

$$Var[I] = \beta_1^2 \sigma_x^2 + \beta_2^2 \sigma_y^2$$

Maintaining Ralph in Alice's employ requires the usual individual rationality constraint be satisfied $CE(a) \geq \overline{CE}$ (normalized to zero). Since Alice desires input H from Ralph, supply of incentives requires the usual incentive compatibility constraint $CE(H) \geq CE(L)$, and task balance requires an additional

[1] It's noteworthy that Ralph's risk aversion (reflected in $\frac{1}{2}\rho Var[I]$) is independent of his input choice.

incentive compatibility constraint $a_1 = a_2 = \frac{H}{2}$.[2] Hence, supply of incentives for balancing tasks requires $\beta_1 + \beta_2 = \beta_1\alpha + \beta_2\gamma$.[3] Putting this together, Alice writes Ralph's input H employment contract to minimize her expected compensation cost subject to the above noted restrictions (for input L or without task balance simply revise the input conditioned on and/or relax the appropriate constraint).

$$\min_{w,\beta_1,\beta_2} w + \beta_1 E[x \mid H] + \beta_2 E[y \mid H]$$
$$s.t.$$

$$\begin{aligned} CE(H) &\geq 0 && (IR) \\ CE(H) &\geq CE(L) && (IC_1) \\ \beta_1 + \beta_2 &= \beta_1\alpha + \beta_2\gamma && (IC_2) \end{aligned}$$

Ralph and Alice recognize $\alpha = 0.7$, $H = 500$, $L = 200$, $\rho = 0.1$, $c_H = 60$, and $c_L = 0$. Notice, incentive compatibility implies $\beta_1 + \beta_2 \geq \frac{c_H - c_L}{H - L}$.

[2] Notice, Ralph's allocation of input between the two tasks desired by Alice cannot exceed his total input. Hence, for input H we have

$$a_1 + a_2 \leq H$$

and for input L we have

$$a_1 + a_2 \leq L$$

[3] Otherwise, Ralph rationally emphasizes the task with the greater return (to maximize his expected utility).

Required:

1. Suppose only performance measure x is available with variance $\sigma_x^2 = 10,000$. To supply incentives for Ralph to provide input H for task one, what contract (w and β_1) does Alice offer Ralph? If Alice wishes Ralph to balance the tasks, why must she settle for input L?

2. Suppose x remains as in 1 and y is available as well with $\gamma = 0.6$ and variance $\sigma_y^2 = 10,000$. To supply incentives for Ralph to provide input H for task one, what contract (w, β_1, and β_2) does Alice offer Ralph? How does Alice's expected compensation cost compare to that in 1? To supply incentives for Ralph to provide a balance of input H for both tasks, what contract (w, β_1, and β_2) does Alice offer Ralph? Compare Alice's expected compensation cost for task balance with the unbalanced tasks.

3. Suppose x remains as in 1 and y is available as well with $\gamma = 0.6$ and variance $\sigma_y^2 = 0$. To supply incentives for Ralph to provide input H for task one, what contract (w, β_1, and β_2) does Alice offer Ralph? How does Alice's expected compensation cost compare to that in 1? To supply incentives for Ralph to provide a balance of input H for both tasks, what contract (w, β_1, and β_2) does Alice offer Ralph? Compare Alice's expected compensation cost for task balance with the unbalanced tasks. In what sense does this setting demonstrate good measures driving out bad measures?

4. Suppose x remains as in 1 except $\sigma_x^2 = 0$ and y is available as well with $\gamma = 1.0$ and variance $\sigma_y^2 = 10,000$. To supply incentives for Ralph to provide input H for task one, what contract (w, β_1, and β_2) does Alice offer Ralph? How does Alice's expected compensation cost compare to that in 1? To supply incentives for Ralph to provide a balance of input H for both tasks, what contract (w, β_1, and β_2) does Alice offer Ralph? Compare Alice's expected compensation cost for task balance with the unbalanced tasks. In what sense does this setting demonstrate bad measures driving out good measures?