Ralph's Symmetry

The variance-covariance structure of two random variables is described by $V = \begin{bmatrix} Var[v_1] & Cov[v_1,v_2] \\ Cov[v_1,v_2] & Var[v_2] \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$. Ralph is interested in exploring diagonalization of V.

A. Cholesky decomposition.

Suggested:

1. Find G such that $GG^T = V$ where G is lower triangular. Recall for a symmetric matrix V can be written as LDL^T where D is a diagonal matrix containing the pivots of V. In other words, $LD^{1/2}\, D^{1/2}L^T = GG^T$.

2. Find $G^{-1} = D^{-1/2}\, L^{-1}$ such that $V(G^T)^{-1}G^{-1} = I$ where G^{-1} is lower triangular.

B. Spectral decomposition.

Suggested:

1. Find Z and M such that $ZMZ^T = V$ where Z contains the orthonormal eigenvectors of V and M is a diagonal matrix of the corresponding eigenvalues of V.

2. Write $ZMZ^T = WW^T = V$ where $W = ZM^{1/2}$. Find W^{-1} where $W^{-1}V(W^T)^{-1} = I$.

3. Compare the product of the pivots from part A with the product of the eigenvalues in part B. Compare these quantities with the determinant of V. Compare the trace of V (sum of the main diagonal elements) with the sum of the eigenvalues.

C. Simultaneous diagonalizability (commutativity)

Suggested:

1. Replace M above with some other eigenvalues, say the pivots D. Create a new matrix $U = ZDZ^T$. Compute $VU - UV$. What do you observe?

2. Why does U commute with V?

3. Why does the identity matrix I commute with any other matrix?