Ralph's Smoothing Incentives

Alice owns a firm (production technology, resources, trade relations, etc.) and employs Ralph to manage these assets. Alice wishes to maximize long-run wealth and pass her wealth along to future generations. She recognizes that most individuals, including Ralph, suffer a short-term horizon so aligning her interests with Ralph's incentives may be challenging. Alice knows a Kelly investment strategy implements her plans but that spanning (or otherwise coping with) a large number of states calls for serious managerial talent. Alice's ruminations emphasize managerial focus on project design and complementary information acquisition.

A continuum of states

Suppose there exists a continuum of states and Ralph manages/designs a large (but finite) number of projects (assets). Alice and Ralph recognize, strictly speaking, that spanning cannot be satisfied. However, they recognize that if portfolios of projects (assets) can be designed and managed such that two portfolios are perfectly negatively correlated then these portfolios can be combined to produce a sure and constant payoff (return) in every state (akin to Arrow-Debreu assets). Since there is no variation in payoffs (returns), Alice could align Ralph's incentives with her interests by using a constant or smooth income stream or rate of return to evaluate Ralph's performance.

Ralph designs/manages the projects by knowing/learning their means as well as the variances/covariances so the natural (maximum entropy) probability assignment is a multivariate normal distribution. Suppose Ralph identifies (designs) two portfolios with mean returns $\mu = \begin{bmatrix} 0.99 \\ 1.05 \end{bmatrix}$ and variance of returns $\Sigma = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$. If Ralph invests $\alpha = \frac{\sigma_2}{\sigma_1 + \sigma_2}$ fraction of assets in the first portfolio and $1 - \alpha$ in the second portfolio where σ_j denotes the standard deviation on portfolio j, then returns are a constant in all states.

Suggested:

1. Determine the constant return, $L\mu$, and verify variance of returns is $L \Sigma L^T = 0$, where $L = \begin{bmatrix} \alpha & 1 - \alpha \end{bmatrix}$.

Complementary information

2. Suppose Ralph can acquire information, y, such that the mean and variance of x and y are $\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} = \begin{bmatrix} 0.99 \\ 1.05 \\ 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix} =$

$\begin{bmatrix} 1 & -2 & 0.4 \\ -2 & 4 & -0.8 \\ 0.4 & -0.8 & 1 \end{bmatrix}$ (in block matrix form). Recall Bayes-Normal theory gives $E[x \mid y = y_0] = \mu_x + \Sigma_{xy}\Sigma_y^{-1}(y_0 - \mu_y)$ and $Var[x \mid y] = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx}$.

a. Determine $E[x \mid y = 0.9]$, $E[x \mid y = 1]$, $E[x \mid y = 1.1]$, and $Var[x \mid y]$.

b. Determine L_y such that $L_y Var[x \mid y] L_y^T = 0$.

c. Does $L_y E[x \mid y = 0.9]$, $L_y E[x \mid y = 1]$, or $L_y E[x \mid y = 1.1]$ deviate from the return in 1?

Complementary project design and information

3. Suppose Ralph can design two sets of portfolios and acquire information y with the following properties: $\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.02 \\ 0.99 \\ 1.03 \\ 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix} =$ $\begin{bmatrix} 1 & -0.28 & -0.36 & -0.36 & 0.6 \\ -0.28 & 1 & -0.36 & -0.36 & 0.6 \\ -0.36 & -0.36 & 1 & -.28 & -0.6 \\ -0.36 & -0.36 & -.28 & 1 & -0.6 \\ 0.6 & 0.6 & -0.6 & -0.6 & 1 \end{bmatrix}$ where the first (second) set of returns are indicated by the first and second (third and fourth) elements.

a. Determine $E[x \mid y = 0.9]$, $E[x \mid y = 1]$, $E[x \mid y = 1.1]$, and $Var[x \mid y]$. Determine $A = \begin{bmatrix} \alpha_1 & 1-\alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1-\alpha_2 \end{bmatrix}$ such that $A Var[x \mid y] A^T = 0$.

b. Determine weight (fraction of assets invested), w, in the two portfolios such that $\begin{bmatrix} w & 1-w \end{bmatrix} E[x \mid y = y_0]$ is maximized for $y_0 = 0.9, 1, 1.1$.

c. Determine weight (fraction of assets invested), w_c, in the two portfolios such that $\begin{bmatrix} w_c & 1-w_c \end{bmatrix} E[x \mid y]$ is constant.

4. Why might Alice motivate a smooth report instead of maximizing

$$\begin{bmatrix} w & 1-w \end{bmatrix} E[x \mid y = y_0]?$$

(hint: how does Alice distinguish long-run from short-run wealth maximization?)