# Ralph's Random Probability Assignment

Ralph is puzzling over how to revise beliefs for two settings which appear to share similar prior beliefs and where evidence is quantified via similar likelihood functions but belief revision may differ substantially. For example, Ralph believes (prior to collecting additional evidence) that the proposition that life has ever existed on Mars is 50 : 50 just like a "fair" coin produces 50 : 50 heads:tails. However, Ralph believes new evidence is unlikely to substantially change his assessment of the coin's fairness but that any evidence of life on Mars may drastically change his beliefs regarding existence of life on Mars.

Ralph proceeds in Bayesian fashion to assign a probability distribution (density function) to the probability associated with each proposition ($\Pr(\text{fair coin})$ and $\Pr(\text{life on Mars})$). Ralph reasons the probability assignment for the fair coin should exhibit a mean equal to $\frac{1}{2}$ and be tightly clustered around the mean. On the other hand, while the probability assignment to life on Mars also has mean equal to $\frac{1}{2}$ the distribution is much more diffuse. Ralph utilizes the maximum entropy principle (maxent) to assign probabilities and quantifies his background knowledge regarding the two propositions in the form of the following moment conditions.

$$\text{fair coin:} \quad E[\log p] = E[\log(1-p)] = -0.6956534$$
$$\text{or geometric mean} = \exp(-0.6956534) \approx 0.4987484$$

$$\text{life on Mars:} \quad E[\log p] = E[\log(1-p)] = -1$$
$$\text{or geometric mean} = \exp(-1) = \frac{1}{e} \approx 0.3678794$$

Suggested:

1. Utilize maxent to assign a probability distribution to each proposition.

— the kernel for each is $p^{-\lambda_1}(1-p)^{-\lambda_2}$ or a kernel for a $Beta(\alpha = 1-\lambda_1, \beta = 1-\lambda_2)$ distribution $f(p)$ with $E[p] = \frac{\alpha}{\alpha+\beta}$. employ maxent to explain the kernel.

— $\alpha$ and $\beta$ can be written in terms of the digamma function $\frac{\partial \log \Gamma(z)}{\partial z}$ such that $\frac{\partial \log \Gamma(\alpha)}{\partial \alpha} - \frac{\partial \log \Gamma(\alpha+\beta)}{\partial(\alpha+\beta)} = E[\log p]$ and $\frac{\partial \log \Gamma(\beta)}{\partial \beta} - \frac{\partial \log \Gamma(\alpha+\beta)}{\partial(\alpha+\beta)} = E[\log(1-p)]$. complete the probability assignment by solving for $\alpha$ and $\beta$. (hint: $R$ contains the digamma function.)

— plot and compare the two density functions assigned to the propositions.

2. Suppose 3 bits of new evidence for each proposition arrive all indicating support. Ralph assigns a binomial distribution as the likelihood function for this data (see Ralph's binomial probability assignment) with kernel $p^s(1-p)^{n-s}$ where $s$ is the number of results consistent with the proposition out of $n$ tests. Determine Ralph's updated probability beliefs for each proposition.

— updated beliefs for each proposition have a $Beta\left(\alpha+s, \beta+n-s\right)$ distribution $f\left(p \mid E\right)$. explain by deriving the kernel for $f\left(p \mid E\right)$.

— determine $E\left[p \mid E\right]$ for each proposition.

— plot and compare the two density functions conditional on the evidence assigned to the propositions.