Ralph's Probability Assignment

Ralph manages a production process that produces red (R), yellow (Y), and green (G) widgets. Daily production is limited to 5 units of only one type of widget. The key to Ralph's production decision is uncertain daily demand for each widget which ranges from 0 to 20 units per day. Ralph assigns probabilities to describe the uncertainty of daily demand consistent with his background knowledge but such that only what he knows is reflected in his probability assessments. In other words, Ralph assigns probabilities so that entropy[1] (uncertainty) is maximized subject to constraints implied by Ralph's background knowledge

$$\begin{array}{l} \underset{p_i \geq 0}{Max} - \sum_{i=0}^{20} p_i \log(p_i) \\ s.t. \\ \quad \sum_{i=0}^{20} p_i = 1 \\ \text{additional constraints} \end{array}$$

Ralph makes daily production decisions to minimize expected lost sales. Let D_R be daily demand for red widgets, D_Y be daily demand for yellow widgets, and D_G be daily demand for green widgets, expected loss if red widgets are produced today is

$$\begin{aligned} &\sum_{i=0}^{20} \Pr(D_R = i) \max(i - S_R - 5, 0) \\ &+ \sum_{i=0}^{20} \Pr(D_Y = i) \max(i - S_Y, 0) \\ &+ \sum_{i=0}^{20} \Pr(D_G = i) \max(i - S_G, 0) \end{aligned}$$

where S_R, S_Y, and S_G represent current stocks of the red, yellow, and green widgets. Expected values of loss for production of yellow and green widgets

[1] Shannon derived a measure of entropy so that five conditions are satisfied: (1) a measure H exists, (2) the measure is smooth, (3) the measure is monotonically increasing in uncertainty, (4) the measure is consistent in the sense that if different measures exist they lead to the same conclusions, and (5) the measure is additive. The measure implied by the five conditions is

$$H = -\sum_{i=1}^{n} p_i \log(p_i)$$

Additivity implies

$$H(x, y) = H(y) + \Pr(y_1) H(x \mid y_1) + \cdots + \Pr(y_n) H(x \mid y_n)$$

In other words, joint entropy equals the entropy of the signals (y) plus the probability weighted average of entropy conditional on the signals. This latter term, $\Pr(y_1) H(x \mid y_1) + \cdots + \Pr(y_n) H(x \mid y_n)$, is called conditional entropy. If you will, this condition is the glue that holds ideas together similar to how Bayes' theorem works for probability theory.

are analogous where production of 5 units is assigned to yellow and green, respectively.

Required:

1. Suppose Ralph only knows the current stock of widgets:

$$\begin{aligned} S_R &= 10 \\ S_Y &= 15 \\ S_G &= 5 \end{aligned}$$

What is the maximum entropy probability assignment for demand of each widget (since Ralph knows nothing regarding cross dependencies of demand, the maximum entropy assignment is that demand for the three widgets is independent).
Based on this probability assignment, which widget will Ralph produce? Does this decision match your intuition?

2. Suppose in addition to Ralph's knowledge of the current stock of widgets Ralph knows the average daily demand for each widget:

$$\begin{aligned} \mu_R &= 5 \\ \mu_Y &= 10 \\ \mu_G &= 1 \end{aligned}$$

Hence, the added constraint for red widgets is

$$\sum_{i=0}^{20} i p_i = 5$$

The added constraint for the other widgets is analogous. What is the maximum entropy probability assignment for daily demand of each widget.
Based on this probability assignment, which widget will Ralph produce? Does this decision match your intuition?