Ralph's Normal Fallacy

Ralph is faced with two decisions and can acquire either of two, both, or no costly information systems. Each decision entails making a point estimate, $\widehat{x}_j$, regarding the eventual value of an outcome, x_j, where $j = 1$ or 2. Ralph's knowledge of the first variable is

$$x_1 = \mu_1 + k\varepsilon_1 + \varepsilon_2$$

and the second variable is

$$x_2 = \mu_2 + \varepsilon_1 + \varepsilon_2$$

where μ_1, μ_2, and k are known constants and ε_1 and ε_2 are mean zero normal random variables with variances, σ_1^2 and σ_2^2, respectively. The first information system reports ε_1 at cost c_1 while the second information system reports ε_2 at cost c_2. For simplicity, the cost of decision error is $E\left[\left(\sum_{j=1}^{2} x_j - \widehat{x}_j\right)^2\right]$ where $\widehat{x}_j$ is our best guess for x_j given the information for decision $j = 1$ or 2. Hence, if both information sources are acquired, perfect prediction is possible and the cost is $c_1 + c_2$. While if neither information sources are acquired, the cost is

$$L^T \begin{bmatrix} k^2\sigma_1^2 + \sigma_2^2 + 2k\sigma_{12} & k\sigma_1^2 + \sigma_2^2 + (1+k)\,\sigma_{12} & k\sigma_1^2 + \sigma_{12} & k\sigma_{12} + \sigma_2^2 \\ k\sigma_1^2 + \sigma_2^2 + (1+k)\,\sigma_{12} & \sigma_1^2 + \sigma_2^2 + 2\sigma_{12} & \sigma_1^2 + \sigma_{12} & \sigma_{12} + \sigma_2^2 \\ k\sigma_1^2 + \sigma_{12} & \sigma_1^2 + \sigma_{12} & \sigma_1^2 & \sigma_{12} \\ k\sigma_{12} + \sigma_2^2 & \sigma_{12} + \sigma_2^2 & \sigma_{12} & \sigma_2^2 \end{bmatrix} L$$

$$\begin{aligned} &= k^2\sigma_1^2 + \sigma_2^2 + 2k\sigma_{12} + \sigma_1^2 + \sigma_2^2 + 2\sigma_{12} + 2\left(k\sigma_1^2 + \sigma_2^2 + (1+k)\,\sigma_{12}\right) \\ &= (k+1)^2\,\sigma_1^2 + 4\sigma_2^2 + 2\,(2+2k)\,\sigma_{12} \end{aligned}$$

where $L^T = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$ is the transpose of L, a column vector, and σ_{12} refers to the covariance between ε_1 and ε_2. Since ε_1 and ε_2 are normally distributed random variables so are x_1 and x_2. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, Ralph knows

$$\begin{aligned} E\left[x \mid \varepsilon_1 = e_1\right] &= \begin{bmatrix} \mu_1 + ke_1 + \frac{\sigma_{12}}{\sigma_1^2}e_1 \\ \mu_2 + e_1 + \frac{\sigma_{12}}{\sigma_1^2}e_1 \end{bmatrix} \\ Var\left[x \mid \varepsilon_1\right] &= \begin{bmatrix} \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} & \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \\ \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} & \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \end{bmatrix} \end{aligned}$$

Hence, the cost of decision error given ε_1 is acquired is

$$\begin{aligned} & c_1 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} & \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \\ \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} & \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ = \; & c_1 + 4\left(\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}\right) \end{aligned}$$

$$
\begin{aligned}
E\left[x \mid \varepsilon_2 = e_2\right] &= \begin{bmatrix} \mu_1 + e_2 + k\frac{\sigma_{12}}{\sigma_2^2}e_2 \\ \mu_2 + e_2 + \frac{\sigma_{12}}{\sigma_2^2}e_2 \end{bmatrix} \\
Var\left[x \mid \varepsilon_2\right] &= \begin{bmatrix} k^2\left(\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right) & k\left(\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right) \\ k\left(\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right) & \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2} \end{bmatrix}
\end{aligned}
$$

Hence, the cost of decision error given ε_2 is acquired is

$$
\begin{aligned}
& c_2 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} k^2\left(\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right) & k\left(\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right) \\ k\left(\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right) & \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
= \; & c_2 + (k+1)^2\left(\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right)
\end{aligned}
$$

Consider the following six cases.

case	k	σ_1^2	σ_2^2	σ_{12}	c_1	c_2
1	99	1	1	0	10	10
2	1	9	9	0	10	10
3	0	9	9	0.9	5	20
4	0	9	9	9	12	15
5	1	9	9	9	7	8
6	1	9	9	0	40	40

Required:

1. For cases 1-6, what information systems, if any, are acquired and what is the total (information and decision error) cost considering both information systems and their full decision implications?

2. What does this example suggest about viewing accounting choice in isolation (of other information and other decisions)?