Ralph's Long-run Frame

Ralph manages a firm and wishes to structure operations to maximize long-run wealth. As a student of information science, Ralph knows expected long-run wealth is maximized via the Kelly criterion. That is, maximize expected compound (or geometric mean) return on investment

$$\max_{w} \quad G[r] = \prod_{j=1}^{n} \left(\sum_{i=1}^{m} w_i r_{ij}\right)^{p_j}$$
$$s.t. \quad \sum_{i=1}^{m} w_i = 1$$

or equivalently maximize the arithmetic mean (expected value) of the natural logarithm of returns

$$\max_{w} \quad E[r] = \sum_{j=1}^{n} p_j \ln\left(\sum_{i=1}^{m} w_i r_{ij}\right)$$
$$s.t. \quad \sum_{i=1}^{m} w_i = 1$$

where w_i is portion of wealth invested in asset i (this quantity may be negative which translates into borrowing against its future payoff, that is borrow the investment amount and return the payoff to the lender), r_{ij} is return (payoff on investment equal to one) on asset i in state j (so that $\sum_{i=1}^{m} w_i r_{ij}$ is the return on the portfolio of assets in state j), and p_j is the probability Ralph assigns to state j. Ralph notes $G[r] = \exp(E[r])$ is a consistency check on his analysis.

In addition, Ralph knows analysis of the problem is greatly simplified by converting nominal assets into Arrow-Debreu investments (assets that payoff in exactly one state and zero in all other states) which draws from no arbitrage and scalable investments that span the states. Let A denote a matrix of returns (on normalized to unity investments) where the rows indicate the asset and the columns indicate the states, v denote a vector of (normalized) investment costs/prices associated with the assets and y denote a vector of Arrow-Debreu (or state) values.

$$Ay = v$$

If A is full rank ($n \times n$ and comprised of linearly independent rows and columns), then

$$y = A^{-1}v$$

and the elements of each row of A^{-1} identifies the portfolio weights on the nominal assets for constructing the Arrow-Debreu investments where $A^{-1}A = I$ (the identity matrix). As the investment cost for each Arrow-Debreu portfolio implied by these weights is y_j the return on Arrow-Debreu state j portfolio is $\frac{1}{y_j}$. The optimal fraction of wealth, k, invested in each Arrow-Debreu portfolio of assets is

$$\max_{k} \quad G[r] = \prod_{j=1}^{n} \left(k_j \frac{1}{y_j}\right)^{p_j}$$
$$s.t. \quad \sum_{j=1}^{n} k_j = 1$$

or

$$\max_{k} \quad E[r] = \textstyle\sum_{j=1}^{n} p_j \ln\left(k_j \frac{1}{y_j}\right)$$
$$s.t. \quad \textstyle\sum_{j=1}^{n} k_j = 1$$

The first order conditions for the Lagrangian

$$\mathcal{L} = \sum_{j=1}^{n} p_j \ln\left(k_j \frac{1}{y_j}\right) - \lambda\left(\sum_{j=1}^{n} k - 1\right)$$

are

$$\tfrac{p_j}{k_j} - \lambda = 0, \quad \text{for all } j$$

Since $\sum k_j = 1 = \sum \frac{p_j}{\lambda} = \frac{1}{\lambda}$, $\lambda = 1$ and $k_j = p_j$. In other words, probability assignment to state j identifies the optimal fractional investment in state j (notice there are no negative investments in Arrow-Debreu portfolios of assets and the optimal weight doesn't depend on the payoff). Ralph recognizes he can never fully deplete his asset base with this investment policy as some fraction of wealth is invested in the state that pays off.

Ralph recognizes his frame represents his state of knowledge (not objective truth which quantum mechanics and conservation of information tells him is much too complex to address in any meaningful time frame). The foregoing discussion yields the key objectives of Ralph's management practice. Ralph undertakes small-scale operational experiments aimed at maximizing long-run wealth to (i) identify a collection of assets that "span" the state space (assets whose payoffs make A full rank, in other words, choose $m = n$ assets including engineered (securitized) contracts with linearly independent payoffs), (ii) acquire information to better exploit opportunities presented by the identified assets (information is the source for synergy in an uncertain world), (iii) rebalance the firm's asset portfolio (by expanding and contracting individual assets), and (iv) continue to explore asset/information synergy opportunities in view of state space considerations along the lines of (i)-(iii).

Ralph knows the Kelly criterion connects to Shannon's noisy channel theorem by equating mutual information,[1] $I(s;z) = H(s) + H(z) - H(s,z)$, with the expected gain (in returns) due to information, $E[r \mid z] - E[r]$, where $H(\cdot) = -\sum_{j=1}^{n} p_j \ln p_j$, or entropy.

First, consider expected gains from information, z, $E[r \mid z] - E[r]$.

$$\begin{aligned}
E[r \mid z] &= \sum_{j=1}^{n} \Pr(z_j) E[r \mid z_j] \\
&= \sum_{j=1}^{n} \Pr(z_j) \sum_{i=1}^{n} \Pr(s_i \mid z_j) \log \frac{\Pr(s_i \mid z_j)}{y_i} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_i, z_j) \log \frac{\Pr(s_i \mid z_j)}{y_i} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_i, z_j) \log \Pr(s_i \mid z_j) - \sum_{i=1}^{n} \Pr(s_i) \log y_i
\end{aligned}$$

$$\begin{aligned}
E[r] &= \sum_{i=1}^{n} \Pr(s_i) \log \frac{\Pr(s_i)}{y_i} \\
&= \sum_{i=1}^{n} \Pr(s_i) \log \Pr(s_i) - \sum_{i=1}^{n} \Pr(s_i) \log y_i
\end{aligned}$$

[1] Mutual information is usually defined as

$$\begin{aligned}
I(X;Y) &= H(Y) - H(Y \mid X) \\
&= H(X) - H(X \mid Y)
\end{aligned}$$

but the additivity axiom

$$\begin{aligned}
H(X,Y) &= H(Y) + H(X \mid Y) \\
&= H(X) + H(Y \mid X)
\end{aligned}$$

allows a form that is often computationally simpler. Substitute

$$H(X \mid Y) = H(X,Y) - H(Y)$$

into the expression for mutual information

$$\begin{aligned}
I(X;Y) &= H(X) - (H(X,Y) - H(Y)) \\
&= H(X) + H(Y) - H(X,Y)
\end{aligned}$$

and

$$\begin{aligned}
E\left[r \mid z\right] - E\left[r\right] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\Pr\left(s_i, z_j\right)\log\Pr\left(s_i \mid z_j\right) - \sum_{i=1}^{n}\Pr\left(s_i\right)\log y_i \\
&\quad - \left(\sum_{i=1}^{n}\Pr\left(s_i\right)\log\Pr\left(s_i\right) - \sum_{i=1}^{n}\Pr\left(s_i\right)\log y_i\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\Pr\left(s_i, z_j\right)\log\Pr\left(s_i \mid z_j\right) - \sum_{i=1}^{n}\Pr\left(s_i\right)\log\Pr\left(s_i\right)
\end{aligned}$$

On the other hand,

$$I\left(s; z\right) = H\left(s\right) + H\left(z\right) - H\left(s, z\right)$$

where

$$H\left(s\right) = -\sum_{i=1}^{n}\Pr\left(s_i\right)\log\Pr\left(s_i\right)$$

$$H\left(z\right) = -\sum_{j=1}^{n}\Pr\left(z_j\right)\log\Pr\left(z_j\right)$$

and

$$\begin{aligned}
H\left(s, z\right) &= -\sum_{i=1}^{n}\sum_{j=1}^{n}\Pr\left(s_i, z_j\right)\log\Pr\left(s_i, z_j\right) \\
&= -\sum_{i=1}^{n}\sum_{j=1}^{n}\Pr\left(s_i, z_j\right)\left\{\log\Pr\left(s_i \mid z_j\right) + \log\Pr\left(z_j\right)\right\} \\
&= -\sum_{i=1}^{n}\sum_{j=1}^{n}\Pr\left(s_i, z_j\right)\log\Pr\left(s_i \mid z_j\right) - \sum_{j=1}^{n}\Pr\left(z_j\right)\log\Pr\left(z_j\right)
\end{aligned}$$

Then,

$$\begin{aligned}
I\left(s; z\right) &= -\sum_{i=1}^{n}\Pr\left(s_i\right)\log\Pr\left(s_i\right) - \sum_{j=1}^{n}\Pr\left(z_j\right)\log\Pr\left(z_j\right) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\Pr\left(s_i, z_j\right)\log\Pr\left(s_i \mid z_j\right) + \sum_{j=1}^{n}\Pr\left(z_j\right)\log\Pr\left(z_j\right) \\
&= -\sum_{i=1}^{n}\Pr\left(s_i\right)\log\Pr\left(s_i\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\Pr\left(s_i, z_j\right)\log\Pr\left(s_i \mid z_j\right)
\end{aligned}$$

Hence,

$$I\left(s; z\right) = E\left[r \mid z\right] - E\left[r\right]$$

Part A.

Suggested:

Suppose Ralph is managing projects involving the following assets and returns denoted by A where returns on asset i in state j are reported in row i, column j, and v denotes the (normalized) price of the assets.

$$A = \begin{bmatrix} & s_1 & s_2 \\ asset_1 & 0.9 & 1.1 \\ asset_2 & 1.1 & 0.9 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1. Solve $Ay = v$ for $y > 0$, a vector where element y_j denotes the price per unit return in state j.

2. Form the Arrow-Debreu portfolios and determine the expected logarithmic return, $E[r]$, and expected compound return, $G[r] = \exp(E[r])$.

$$E[r] = p^T \ln(\Omega k)$$

where p is a vector of assigned state probabilities, Ω is a diagonal matrix with returns on Arrow-Debreu portfolios, $\frac{1}{y_j}$, along the main diagonal, and k is a vector of the fraction of wealth invested in each Arrow-Debreu portfolio.

Suppose Ralph acquires the following information.

$\Pr(s,z)$	s_1	s_2	$\Pr(z)$
z_1	$\frac{1}{2}$	0	$\frac{1}{2}$
z_2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\Pr(s)$	$\frac{1}{2}$	$\frac{1}{2}$	

As uncertainty is (partially) resolved Ralph rebalances his portfolio of projects by expanding and contracting investment (recall the optimal investment strategy matches the investment in each state with the likelihood assigned to the state).

3. Verify that mutual information, $I(s;z) = H(s) + H(z) - H(s,z)$, equals the expected gain from the information (value of the information to Ralph), $E[gain] = E[r \mid z] - E[r]$ where $E[r \mid z] = \Pr(z_1) E[r \mid z_1] + \Pr(z_2) E[r \mid z_2]$.

4. Explore the role of spanning and scalability by comparing the composition of the Arrow-Debreu portfolios conditional on the signals, $w = \left(A^{-1}\right)^T \Omega k$, and expected compound return, $G[r]$, with that where spanning and scalability are ignored, that is, where the fraction of wealth invested in any asset is less than or equal to one, $0 \leq w_i \leq 1$.

5. Is the information worth Ralph pursuing? How much does usage (scalability) of the information impact Ralph's value of the information? How important is Ralph's commitment to nurturing reputation and long-term relations with trading partners? Does reputation impact scalability?

6. For the optimal (Kelly) portfolios, what is the likelihood the firm goes bankrupt — the sum of state probabilities involving a zero (or negative) return? How does maximizing compound returns (or the geometric mean) avoid "gambler's ruin"?

7. Alternatively, suppose Ralph maximizes expected nominal (short-run) portfolio return (again the portfolio weights sum to one). What is the likelihood the firm goes bankrupt given this alternative objective function? What, if anything, does this suggest about recent failures resulting in bail-outs?

Part B.

Alternatively, suppose Ralph believes he operates in a four-state world and he identifies the following collection of assets and corresponding returns.

$$A = \begin{bmatrix} & s_1 & s_2 & s_3 & s_4 \\ asset_1 & 1 & 1 & 1 & 1 \\ asset_2 & 1.10 & \frac{1}{1.10} & \frac{1.09}{1.10} & 1 \\ asset_3 & 1 & \frac{1.09}{1.10} & 1.10 & \frac{1}{1.10} \\ asset_4 & 1 & 1.10 & \frac{1.09}{1.10} & \frac{1}{1.10} \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and Ralph's operations lead him to acquire information and assign the following joint probabilities

	s_1	s_2	s_3	s_4	$\Pr(z)$
z_1	0.20	0.20	0.05	0.05	$\frac{1}{2}$
z_2	0.05	0.05	0.20	0.20	$\frac{1}{2}$
$\Pr(s)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

Suggested: Repeat the questions from part A for this scenario.