Ralph's long-run quantum accounting

Ralph makes production-investment decisions to maximize long-run wealth (or equivalently, maximize expected log returns). By the law of large numbers, accounting rate of return converges to expected log return, $E[r]$.

$$\ln\left(1 + \frac{income}{assets}\right) \longrightarrow E[r]$$

In a world of inherent uncertainty, quantum information describes the properties of a system as residing in superposition (a sort of simultaneous existence in all possible states) until measurement reveals a specific property. Superposition is conveyed via qubits (quantum bits) such as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha^2 + \beta^2 = 1$, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $|\psi\rangle$ is called ket and $\langle\psi|$ is bra, where bra is the conjugate-transpose of ket, a row vector. Hence, $\langle\psi_i|\psi_j\rangle$ bra-ket is the vector inner product.

Ralph's Kelly-Ross maximum entropy probability assignment algorithm applies to incomplete as well as complete market settings (at least, so long as a riskless asset exists). The algorithm follows.

1. For a state-act-outcome matrix A (in return form) with states by columns and assets by rows, solve $Ay^p = v$ for y^p any consistent solution (not necessarily unique) where v is ι a vector of ones (normalized asset prices).

2. Solve $AN^T = 0$ for N (if any exists other than zero), the nullspace of A.

3. Let $y = y^p + N^T k$.

4. $max\ h(y^p) = -(y^p)^T \ln y^p$ by choosing $k = k*$.

5. Let $y^k = (y^p \mid k = k*)$ and $f = y^k / \left(y^k \iota\right)$ where ι is a vector of ones.

6. Let $\Omega = \begin{bmatrix} \frac{1}{y_1^k} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{y_n^k} \end{bmatrix}$. Find expected log returns given perfect information.

$$E[r \mid PI] = f^T \ln(\Omega\iota)$$

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7. Find expected log returns without information.

$$E[r] = f^T \ln(\Omega f) \tag{1}$$

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8. Verify $E[r \mid PI] = E[r] + h(f)$.

9. Create a $t \times t$ quantum density operator $\rho = \sum_i y_i^{zk} |\psi_i\rangle \langle\psi_i| = Q\Lambda Q^T$ where Λ is a diagonal matrix with the eigenvalues λ along its main diagonal and Q is a matrix of the corresponding orthonormal eigenvectors.

10. Create an accounting observable $O_q = Q \begin{bmatrix} \ln(\alpha\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \ln(\alpha\lambda_t) \end{bmatrix} Q^T$.

11. Solve $E[r^q] \equiv Tr[\rho O_q] = E[r]$ for numeraire α where $Tr[\,]$ refers to the trace or sum of diagonal elements of a matrix.

12. Find expected log returns with perfect information for the quantum transformed production system.

$$E[r^q \mid PI] = Tr[\rho O_p]$$

where $O_p = Q \begin{bmatrix} \ln\alpha & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \ln\alpha \end{bmatrix} Q^T$.

13. Compute quantum (von Neumann) entropy $s(\rho) = -\lambda^T \ln \lambda$.

14. Verify $E[r^q \mid PI] = E[r^q] + s(\rho)$.

Complete economy

Case 1:

state	$\vert 0\rangle$	$\vert 1\rangle$
probability	0.5	0.5
$invest_1$	1	1
$invest_2$	$\frac{1}{2}$	$\frac{3}{2}$

Incomplete economy

Case 2 — two nonorthogonal component ensemble:

state	$\vert 0\rangle$	$\vert \psi_1\rangle$
probability	0.5	0.5
$invest_1$	1	1
$invest_2$	$\frac{1}{2}$	$\frac{3}{2}$

where $|\psi_1\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ ($|\psi_1\rangle$ indicates the probability of $|0\rangle$ is only $\frac{9}{25}$ and the probability of its complement $|1\rangle$ is $\frac{16}{25}$).

Case 3 — create riskless asset from two nonorthogonal component ensemble:

state	$\vert 0\rangle$	$\vert \psi_1\rangle$
probability	0.5	0.5
$invest_1$	1.05	0.95
$invest_2$	$\frac{1}{2}$	$\frac{3}{2}$

case 4 — three component ensemble:

state	$\vert 0\rangle$	$\vert \psi_1\rangle$	$\vert \psi_2\rangle$
probability	0.3	0.5	0.2
$invest_1$	1	1	1
$invest_2$	1.45099	0.129403	2.5

where $|\psi_2\rangle = \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ($|\psi_2\rangle$ indicates the probability of $|0\rangle$ is only $\frac{1}{10}$ and the probability of its complement $|1\rangle$ is $\frac{9}{10}$).

Case 5 — create riskless asset from three component ensemble:

state	$\vert 0\rangle$	$\vert 1\rangle$	$\vert +\rangle$
probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$invest_1$	1.071	1.02	0.969
$invest_2$	0.51	1.02	1.53

where $|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$. This suggests the first two states are mutually exclusive but the third state is completely indistinguishable from either of the first two states.

Part A

Suggested:

1. Determine the riskless return in each case? (hint: find the weights on the assets that produce a constant payoff in all states; the weights sum to one. If you are unable to find the riskless return utilize steps 1-5 of the Kelly Ross maximum entropy algorithm to find y^k and proceed.)

2. For cases 1 through 5, apply steps 9-14 of Ralph's Kelly-Ross maximum entropy algorithm to identify expected log returns with perfect information and without information for the classical frame (you might find steps 1-8 helpful) and the quantum frame.

3. Compare the classical frame with the quantum frame based on expected log returns without information as well as based on entropy.

Part B

Suppose Ralph has the opportunity to acquire information x for cases 1 through 3 as follows where $|\psi_1\rangle = |1\rangle$ for case 1.

$\Pr(\|\psi_i\rangle, x_j)$	$\|\psi_0\rangle = \|0\rangle$	$\|\psi_1\rangle$	$\Pr(x)$
x_1	$\frac{1}{4}$	0	$\frac{1}{4}$
x_2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$\Pr(\|\psi_i\rangle)$	$\frac{1}{2}$	$\frac{1}{2}$	

Information x for case 4 is

$\Pr(\|\psi_i\rangle, x_j)$	$\|\psi_0\rangle = \|0\rangle$	$\|\psi_1\rangle$	$\|\psi_2\rangle$	$\Pr(x)$
x_1	0.2	0.4	0	0.6
x_2	0.1	0.05	0.1	0.25
x_3	0	0.05	0.1	0.15
$\Pr(\|\psi_i\rangle)$	0.3	0.5	0.2	

and information x for case 5 is

$\Pr(\|\psi_i\rangle, x_j)$	$\|\psi_0\rangle = \|0\rangle$	$\|\psi_1\rangle = \|1\rangle$	$\|\psi_2\rangle = \|+\rangle$	$\Pr(x)$
x_1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$
x_2	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$
x_3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
$\Pr(\|\psi_i\rangle)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

Then, a conditional density operator is

$$\rho(x_j) = \sum_i \Pr(|\psi_i\rangle \mid x_j) |\psi_i\rangle \langle\psi_i|$$

$$= Q(x_j) \Lambda(x_j) Q(x_j)^T$$

and conditional (composite) classical/quantum entropy is

$$s(\rho(x)) = \sum_j \Pr(x_j) s(\rho(x_j))$$

where $s(\rho(x_j)) = -\lambda(x_j)^T \ln \lambda(x_j)$.

Further, a conditional accounting observable is

$$O_q(x_j) = Q(xj) \begin{bmatrix} \ln[\alpha\lambda_1(x_j)] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \ln[\alpha\lambda_t(x_j)] \end{bmatrix} Q(x_j)^T$$

Also,

$$E[r \mid x_j] = Tr[\rho(x_j) O_q(x_j)]$$

and

$$E\left[r \mid x\right] = \sum_j \Pr\left(x_j\right) E\left[r \mid x_j\right]$$

The expected gain (in log returns) from information x is

$$E\left[gain \mid x\right] \equiv E\left[r \mid x\right] - E\left[r\right]$$

$$= I\left(\left|\psi_i\right\rangle; x\right) \equiv s\left(\rho\right) - s\left(\rho\left(x\right)\right)$$

$I\left(\left|\psi_i\right\rangle; x\right)$ is (composite) classical/quantum mutual information.

Suggested:

1. Determine $E\left[r \mid x\right]$ and $E\left[gain \mid x\right]$ for cases 1 through 5.
2. Verify $E\left[gain \mid x\right] = I\left(\left|\psi_i\right\rangle; x\right)$ for cases 1 through 5.