Ralph's long-run quantum accounting

Ralph makes production-investment decisions to maximize long-run wealth (or equivalently, maximize expected log returns). By the law of large numbers, accounting rate of return converges to expected log return, $E[r]$.

$$\ln\left(1+\frac{income}{assets}\right) \longrightarrow E[r]$$

In a world of inherent uncertainty, quantum information describes the properties of a system as residing in superposition (a sort of simultaneous existence in all possible states) until measurement reveals a specific property. Superposition is conveyed via qubits (quantum bits) such as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where $\alpha^2+\beta^2=1$, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $|\psi\rangle$ is called ket and $\langle\psi|$ is bra, where bra is the conjugate-transpose of ket, a row vector. Hence, $\langle\psi_i|\psi_j\rangle$ bra-ket is the vector inner product.

Ralph's Kelly-Ross maximum entropy probability assignment algorithm applies to incomplete as well as complete market settings (at least, so long as a riskless asset exists). The algorithm follows.

1. For a state-act-outcome matrix A (in return form) with states by columns and assets by rows, solve $Ay^p = v$ for y^p any consistent solution (not necessarily unique) where v is ι a vector of ones (normalized asset prices).

2. Solve $AN^T = 0$ for N (if any exists other than zero), the nullspace of A.

3. Let $y = y^p + N^T k$.

4. Find $z = y^T \iota$.

5. Create $y^z = \frac{1}{z} y$.

6. $max\ h(y^z) = -(y^z)^T \ln y^z$ by choosing $k = k*$.

7. Let $y^{zk} = (y^z \mid k = k*)$.

8. Let $y^k = (y \mid k = k*)$.

9. Let $\Omega = \begin{bmatrix} \frac{1}{y_1^k} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{y_n^k} \end{bmatrix}$. Find expected log returns given perfect information.

$$E[r \mid PI] = \left(y^{zk}\right)^T \ln(\Omega\iota)$$

10. Find expected log returns without information.

$$E[r] = \left(y^{zk}\right)^T \ln\left(\Omega y^{zk}\right) \quad (1)$$

11. Verify $E[r \mid PI] = E[r] + h\left(y^{zk}\right)$.

12. Create a t X t quantum density operator $\rho = \sum_i y_i^{zk} |\psi_i\rangle \langle\psi_i| = Q\Lambda Q^T$ where Λ is a diagonal matrix with the eigenvalues λ along its main diagonal and Q is a matrix of the corresponding orthonormal eigenvectors.

13. Create an accounting observable $O_q = Q \begin{bmatrix} \ln \frac{\lambda_1}{y^q} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \ln \frac{\lambda_t}{y^q} \end{bmatrix} Q^T$.

14. Solve $E[r^q] \equiv Tr[\rho O_q] = E[r]$ for y^q.

15. Find expected log returns with perfect information for the quantum transformed production system. $E[r^q \mid PI] = Tr[\rho O_p]$ where $O_p = Q \begin{bmatrix} \ln \frac{1}{y^q} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \ln \frac{1}{y^q} \end{bmatrix} Q^T$.

16. Compute quantum (von Neumann) entropy $s(\rho) = -\lambda^T \ln \lambda$.

17. Verify $E[r^q \mid PI] = E[r^q] + s(\rho)$.

Complete economy

Case 1:

state	$\|0\rangle$	$\|1\rangle$
probability	0.5	0.5
$invest_1$	1	1
$invest_2$	$\frac{1}{2}$	$\frac{3}{2}$

Incomplete economy

Case 2 — two nonorthogonal component ensemble:

state	$\|0\rangle$	$\|\psi_1\rangle$
probability	0.5	0.5
$invest_1$	1	1
$invest_2$	$\frac{1}{2}$	$\frac{3}{2}$

where $|\psi_1\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ (given the second state, $|\psi_1\rangle$ indicates the probability of $|0\rangle$ is only $\frac{9}{25}$ and the probability of its complement $|1\rangle$ is $\frac{16}{25}$).

Case 3 — create riskless asset from two nonorthogonal component ensemble:

state	$\|0\rangle$	$\|\psi_1\rangle$
probability	0.5	0.5
$invest_1$	1.05	0.95
$invest_2$	$\frac{1}{2}$	$\frac{3}{2}$

case 4 — three component ensemble:

state	$\|0\rangle$	$\|\psi_1\rangle$	$\|\psi_2\rangle$
probability	0.3	0.5	0.2
$invest_1$	1	1	1
$invest_2$	1.45099	0.129403	2.5

where $|\psi_2\rangle = \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (given the third state, $|\psi_2\rangle$ indicates the probability of $|0\rangle$ is only $\frac{1}{10}$ and the probability of its complement $|1\rangle$ is $\frac{9}{10}$).

Case 5 — create riskless asset from three component ensemble:

state	$\|0\rangle$	$\|1\rangle$	$\|+\rangle$
probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$invest_1$	1.071	1.02	0.969
$invest_2$	0.51	1.02	1.53

where $|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$. This suggests the first two states are mutually exclusive but the third state is completely indistinguishable from either of the first two states.

Part A

Suggested:

1. For cases 1 through 5, apply Ralph's Kelly-Ross maximum entropy algorithm to identify expected log returns with perfect information and without information for the classical frame and the quantum frame. How does y^{zk} compare with the assigned state probabilities?

2. Compare the classical frame with the quantum frame based on expected log returns without information as well as entropy.

3. What is the riskless return in each case? (hint: find the weights on the assets that produce a constant payoff in all states; the weights sum to one.)

Part B

Suppose Ralph has the opportunity to acquire information x for cases 1 through 3 as follows.

$\Pr(\vert\psi_i\rangle, x_j)$	$\vert\psi_0\rangle = \vert 0\rangle$	$\vert\psi_1\rangle$	$\Pr(x)$
x_1	$\frac{1}{4}$	0	$\frac{1}{4}$
x_2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$\Pr(\vert\psi_i\rangle)$	$\frac{1}{2}$	$\frac{1}{2}$	

Information x for case 4 is

$\Pr(\vert\psi_i\rangle, x_j)$	$\vert\psi_0\rangle = \vert 0\rangle$	$\vert\psi_1\rangle$	$\vert\psi_2\rangle$	$\Pr(x)$
x_1	0.2	0.4	0	0.6
x_2	0.1	0.05	0.1	0.25
x_3	0	0.05	0.1	0.15
$\Pr(\vert\psi_i\rangle)$	0.3	0.5	0.2	

and information x for case 5 is

$\Pr(\vert\psi_i\rangle, x_j)$	$\vert\psi_0\rangle = \vert 0\rangle$	$\vert\psi_1\rangle = \vert 1\rangle$	$\vert\psi_2\rangle = \vert +\rangle$	$\Pr(x)$
x_1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$
x_2	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$
x_3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
$\Pr(\vert\psi_i\rangle)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

Then, a conditional density operator is

$$\rho(x_j) = \sum_i \Pr(|\psi_i\rangle \mid x_j) |\psi_i\rangle \langle\psi_i|$$

$$= Q(x_j) \Lambda(x_j) Q(x_j)^T$$

and conditional (composite) classical/quantum entropy is

$$s(\rho(x)) = \sum_j \Pr(x_j) s(\rho(x_j))$$

where $s(\rho(x_j)) = -\lambda(x_j)^T \ln \lambda(x_j)$.

Further, a conditional accounting observable is

$$O_q(x_j) = Q(xj) \begin{bmatrix} \ln \frac{\lambda_1(x_j)}{y^q} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \ln \frac{\lambda_n(x_j)}{y^q} \end{bmatrix} Q(x_j)^T$$

Also,

$$E\left[r \mid x_j\right] = Tr[\rho\left(x_j\right) O_q\left(x_j\right)]$$

and

$$E\left[r \mid x\right] = \sum_j \Pr\left(x_j\right) E\left[r \mid x_j\right]$$

The expected gain (in log returns) from information x is

$$E\left[gain \mid x\right] \equiv E\left[r \mid x\right] - E\left[r\right]$$

$$= I\left(\left|\psi_i\right\rangle ; x\right) \equiv s\left(\rho\right) - s\left(\rho\left(x\right)\right)$$

$I\left(\left|\psi_i\right\rangle ; x\right)$ is (composite) classical/quantum mutual information.

Suggested:

1. Determine $E\left[r \mid x\right]$ and $E\left[gain \mid x\right]$ for cases 1 through 5.
2. Verify $E\left[gain \mid x\right] = I\left(\left|\psi_i\right\rangle ; x\right)$ for cases 1 through 5.