Ralph's long-run equilibrium

Ralph is managing in an economy where every agent maximizes long-run wealth (employs a Kelly strategy) but each agent acquires their own information (information is heterogeneous). There are three equally endowed agents (a, b, and c) in the economy (with initial wealth normalized to one).

Payoffs in initial state i on nominal assets are depicted in the rows and states in the columns of A_i.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0.9 & 1.1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.1 & 1.1 \\ 0.9 & 1.2 \end{bmatrix}$$

Equilibrium state prices y_i satisfy $A_i y_i = x_i$ where x_i is equilibrium price vector for nominal assets.

Each agent acquires information involving two possible signals where z_i refers to agent i's acquired information (Ralph is agent 1). On observing their own signal, the agents rebalance their portfolios. The positions taken by the agents reveal their private information and the agents rebalance to achieve equilibrium.

Agent a's joint probability assignment regarding information $z = abc$ (where signal a is acquired by agent a, signal b by agent b, and signal c by agent c) and transition state from initial state i to state j, s_{ij}, is tabulated below.

z	$\Pr(s_{11}, z)$	$\Pr(s_{12}, z)$	$\Pr(s_{21}, z)$	$\Pr(s_{22}, z)$	$\Pr(z)$
111	0.017307692	0.040384615	0.047115385	0.020192308	0.125
112	0.017307692	0.040384615	0.047115385	0.020192308	0.125
121	0.017307692	0.040384615	0.047115385	0.020192308	0.125
122	0.017307692	0.040384615	0.047115385	0.020192308	0.125
211	0.017307692	0.040384615	0.033653846	0.033653846	0.125
212	0.017307692	0.040384615	0.033653846	0.033653846	0.125
221	0.017307692	0.040384615	0.033653846	0.033653846	0.125
222	0.017307692	0.040384615	0.033653846	0.033653846	0.125
$\Pr(s_{ij})$	0.13846154	0.32307692	0.32307692	0.21538462	
$\Pr(s)$		0.46153846		0.53846154	

Agent a's joint probability distribution

Agent b's joint probability assignment is below.

z	$\Pr(s_{11}, z)$	$\Pr(s_{12}, z)$	$\Pr(s_{21}, z)$	$\Pr(s_{22}, z)$	$\Pr(z)$
111	0.005769231	0.051923077	0.040384615	0.026923077	0.125
112	0.005769231	0.051923077	0.040384615	0.026923077	0.125
121	0.028846154	0.028846154	0.040384615	0.026923077	0.125
122	0.028846154	0.028846154	0.040384615	0.026923077	0.125
211	0.005769231	0.051923077	0.040384615	0.026923077	0.125
212	0.005769231	0.051923077	0.040384615	0.026923077	0.125
221	0.028846154	0.028846154	0.040384615	0.026923077	0.125
222	0.028846154	0.028846154	0.040384615	0.026923077	0.125
$\Pr(s_{ij})$	0.13846154	0.32307692	0.32307692	0.21538462	
$\Pr(s)$		0.46153846		0.53846154	

Agent b's joint probability distribution

Agent c's joint probability assignment is below.

z	$\Pr(s_{11}, z)$	$\Pr(s_{12}, z)$	$\Pr(s_{21}, z)$	$\Pr(s_{22}, z)$	$\Pr(z)$
111	0.028846154	0.028846154	0.033653846	0.033653846	0.125
112	0.005769231	0.051923077	0.047115385	0.020192308	0.125
121	0.028846154	0.028846154	0.033653846	0.033653846	0.125
122	0.005769231	0.051923077	0.047115385	0.020192308	0.125
211	0.028846154	0.028846154	0.033653846	0.033653846	0.125
212	0.005769231	0.051923077	0.047115385	0.020192308	0.125
221	0.028846154	0.028846154	0.033653846	0.033653846	0.125
222	0.005769231	0.051923077	0.047115385	0.020192308	0.125
$\Pr(s_{ij})$	0.13846154	0.32307692	0.32307692	0.21538462	
$\Pr(s)$		0.46153846		0.53846154	

Agent c's joint probability distribution

Suppose the three agents acquire signal $z = 111$ in initial state 1 and $z = 111$ in initial state 2.

Suggested:

1. Verify each agent's revised probability belief in each initial state based on only their own signal and based on all three agent's signals.

$$\begin{aligned}
F^a(z^a = 1) &= F^a(z = 111) = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix} \\
F^b\left(z^b = 1\right) &= F^b(z = 111) = \begin{bmatrix} 0.1 & 0.9 \\ 0.6 & 0.4 \end{bmatrix} \\
F^c(z^c = 1) &= F^c(z = 111) = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}
\end{aligned}$$

2. Verify the representative agent's probability assignment in each initial state.

$$F(z = 111) = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}$$

Hint: The representative agent's probability assignment is the value-weighted, in this case equal-weighted, average of the agents' beliefs in each initial state. This follows from the notion of a representative agent. That is, the representative agent has the same percentage demand for the assets as the aggregate percentage demand for the diverse investors. Hence,

$$w_i^k = \left(A_i^T\right)^{-1} \Omega_i p_i^k \quad \text{for all agents } k$$

where w_i^k is the fractional investment in each nominal asset for agent k in initial state i, A_i is the equilibrium return on assets in initial state i, Ω_i is a diagonal matrix of the equilibrium returns on Arrow-Debreu assets in initial state i, and p_i^k is the fractional wealth invested in each Arrow-Debreu asset equal to agent k's state probability beliefs in initial state i (consistent with a Kelly strategy). Also, the representative investor's demand is

$$w_i^M = \left(A_i^T\right)^{-1} \Omega_i p_i^M$$

where $w_i^M = \sum_k \alpha^k w_i^k$ and α^k is agent k's endowment relative to total endowments. Since spanning and no arbitrage are satisfied, the equilibrium is uniquely defined by A_i and Ω_i. Therefore, $p_i^M = \sum_k \alpha^k p_i^k$. Of course, the result is immediate for Kelly strategy agents if we focus on demand for Arrow-Debreu assets.

3. Use the pricing kernel (from the recovery theorem), $\frac{p_{ij}}{f_{ij}} = \delta \frac{U'(c_j)}{U'(c_i)}$, to determine equilibrium state prices based on the representative agent's probability assignment in each initial state where p_{ij} is the transition state price from initial

state i to state j, f_{ij} is the transition probability, δ is the representative agent's time discount, and $U^{'}(c_k)$ is the marginal utility for consumption in state k. (hint: let $U^{'}(c_1) = \frac{1}{z}$ and $U^{'}(c_2) = 1$ where z is a parameter of the eigenvector from the state price matrix P associated with the maximum eigenvalue δ. Then, solve

$$\begin{aligned} \frac{p_{11}}{f_{11}} &= \delta \\ \frac{p_{22}}{f_{22}} &= \delta \\ \frac{p_{12}}{f_{12}} &= \delta z \\ \frac{p_{21}}{f_{21}} &= \delta \frac{1}{z} \end{aligned}$$

for p_{11}, p_{21}, δ, and z where $p_{12} = \frac{1}{r_{f(1)}} - p_{11} = 1 - p_{11}$ and $p_{22} = \frac{1}{r_{f(2)}} - p_{21} = \frac{1}{1.10} - p_{21}$, $r_{f(i)}$ refers to the risk-free return in initial state i.) Verify $F = \frac{1}{\delta} \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{bmatrix} P \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$ matches the representative agent's state-transition probability assignment.

4. Determine equilibrium prices for the nominal assets, x_i, in each initial state.

5. Determine equilibrium state-transition returns in each initial state, $A_i^* = X_i^{-1} A_i$, where X_i is a diagonal matrix with elements x_i.

6. Verify the steady-state probability distribution associated with $z = 111$.

$$\Pr(s) = \begin{bmatrix} 0.461538 & 0.538462 \end{bmatrix}$$

7. Determine each agent's expected gain from (their own) information or mutual information $I(s; z) = H(s) + H(s_i, z) - H(s_{ij}, z)$ where $H(x) = -\sum_k \Pr(x_k) \ln \Pr(x_k)$ or entropy, $\Pr(s)$ refers to the steady-state probability distribution, $\Pr(s_i, z)$ is the joint initial state and information probability distribution, and $\Pr(s_{ij}, z)$ is the joint state-transition and information probability distribution. (Note: the three agents represent three large pools of individuals with homogeneous beliefs to which each individual makes a negligible contribution to equilibrium.)