Ralph's Implicit Incentives[1]

Ralph, a risk neutral individual, manages two products. Ralph and others who value his services (the market, also risk neutral) understand he impacts production as follows.

$$\begin{aligned} q_1 &= \alpha_1\theta + \beta_1 e + \varepsilon_1 \\ q_2 &= \alpha_2\theta + \beta_2 e + \varepsilon_2 \end{aligned}$$

where q_i is production of product i, θ represents Ralph's permanent skill contribution, e represents Ralph's transitory effort, ε_i represents mean zero normally distributed variation in production (variance σ_ε^2) with precision $\tau_\varepsilon = \frac{1}{\sigma_\varepsilon^2}$, and α_i represents product i's sensitivity to skill while β_i represents product i's sensitivity to effort (for simplicity, α_i and β_i are each normalized to sum to one; $\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = 1$). Production is observable to all parties but not contractible as it is not mutually verifiable. Hence, Ralph does not have a pay-for-performance (explicit incentives) employment contract. Alternatively, Ralph is paid a market wage that reflects his perceived (permanent or skill) value.[2]

However, since production bundles skill and effort contributions, Ralph may try to finesse the market's perception of his skill by increasing effort. The market recognizes such posturing motives and conjectures effort $\widehat{e}$. Given this conjecture, evidence regarding Ralph's skill from observed production is

$$\Im(\theta) = \frac{q_i - \beta_i \widehat{e}}{\alpha_i}$$

Prior to first period production, Ralph's wage, w_1, is equal to the market's prior beliefs (with precision τ_θ) regarding Ralph's skill, $\overline{\theta}$. After production is reported, Ralph's wage, w_2, reflects revised beliefs regarding Ralph's skill. Importantly, production reports can be aggregated, $q_1 + q_2$, or disaggregated, (q_1, q_2). The market's updated expectation of Ralph's skill given aggregate production and conjectured effort, $\widehat{e}$, is

$$E[\theta \mid q_1 + q_2, \widehat{e}] = \frac{\tau_\theta \overline{\theta} + \tau_\varepsilon/2\,(q_1 + q_2 - \widehat{e})}{\tau_\theta + \tau_\varepsilon/2}$$

Similarly, the market's updated expectation of Ralph's skill given disaggregate production and conjectured effort, $\widehat{e}$, is

$$E[\theta \mid q_1, q_2, \widehat{e}] = \frac{\tau_\theta \overline{\theta} + \tau_\varepsilon \alpha_1^2 (q_1 - \beta_1 \widehat{e})/\alpha_1 + \tau_\varepsilon \alpha_2^2 (q_2 - \beta_2 \widehat{e})/\alpha_2}{\tau_\theta + \tau_\varepsilon \alpha_1^2 + \tau_\varepsilon \alpha_2^2}$$

Ralph's expected utility for discounted (at rate δ) expected continuation wages, w_2, based on aggregate production is

$$EU = \delta E[\theta \mid q_1 + q_2, \widehat{e}] - v(e)$$

[1] Adapted from Arya and Mittendorf, "The benefits of aggregate performance metrics in the presence of career concerns."

[2] Recall flat wages are ineffective in supplying incentives for transitory inputs like effort.

where $v(e) = \frac{1}{2}e^2$ represents Ralph's personal cost of effort. Ralph, of course, chooses e and the first order condition for the maximization problem is

$$\begin{aligned} \frac{\delta\tau_\varepsilon/2}{\tau_\theta + \tau_\varepsilon/2} &= \frac{dv(e)}{de} \\ \frac{\delta\tau_\varepsilon/2}{\tau_\theta + \tau_\varepsilon/2} &= e^A \end{aligned}$$

$e^A = \widehat{e}^A$ is the (rational expectations) equilibrium for aggregate production.

Likewise, Ralph's expected utility for discounted expected continuation wages, w_2, based on disaggregate production is

$$EU = \delta E[\theta \mid q_1, q_2, \widehat{e}] - v(e)$$

The first order condition for the disaggregate production maximization problem is

$$\begin{aligned} \frac{\delta\tau_\varepsilon(\alpha_1\beta_1 + \alpha_2\beta_2)}{\tau_\theta + \tau_\varepsilon\alpha_1^2 + \tau_\varepsilon\alpha_2^2} &= \frac{dv(e)}{de} \\ \frac{\delta\tau_\varepsilon(\alpha_1\beta_1 + \alpha_2\beta_2)}{\tau_\theta + \tau_\varepsilon\alpha_1^2 + \tau_\varepsilon\alpha_2^2} &= e^D \end{aligned}$$

$e^D = \widehat{e}^D$ is the (rational expectations) equilibrium for disaggregate production. The sequence of events is summarized in the following timeline.

Ralph is offered an in initial wage, w_1	Ralph exerts effort e	labor market learns either $x_1 + x_2$ or (x_1, x_2)	labor market determines Ralph's continuation wage, w_2

timeline

Suppose common knowledge parameters are

$$\begin{array}{ll} \alpha_1 = \frac{2}{3} & \alpha_2 = \frac{1}{3} \\ \beta_1 = \frac{1}{3} & \beta_2 = \frac{2}{3} \\ \overline{\theta} = \frac{1}{2} & \tau_\theta = \frac{1}{3} \\ \tau_\varepsilon = \frac{1}{2} & \\ \delta = 1 & \end{array}$$

While only Ralph knows his skill is $\theta = \frac{5}{11}$.[3]

[3] You might find it interesting to contrast your results with $\theta = \frac{6}{11}$ or $\theta = \overline{\theta}$.

Required:

1. For aggregate production, find Ralph's equilbrium effort level e^A, expected continuation wage $E\left[\theta \mid q_1 + q_2, \widehat{e}^A\right]$, and EU where $q_1 + q_2 = E[q_1 + q_2 \mid \theta]$.

2. For disaggregate production, find Ralph's equilbrium effort level e^D, expected continuation wage $E\left[\theta \mid q_1, q_2, \widehat{e}^D\right]$, and EU where $q_1 = E[q_1 \mid \theta]$ and $q_2 = E[q_2 \mid \theta]$.

3. Expected surplus is $EU(Ralph) + EU(employer) = \{E[wage] - v(e)\} + \{E[q_1 + q_2] - E[wage]\} = E[q_1 + q_2] - v(e)$.[4] Is aggregate or disaggregate production more efficient (greater expected surplus)? For which features of this setting might accounting play a role?

4. Suppose $\alpha_1 = \beta_1 = \frac{2}{3}$ and $\alpha_2 = \beta_2 = \frac{1}{3}$, repeat questions 1-3.

[4] First-best marginal expected surplus in effort is

$$\begin{aligned} \frac{dE[q_1 + q_2]}{de} &= \frac{dv(e)}{de} \\ 1 &= e \end{aligned}$$

Appendix

Here we illustrate belief updating with one signal — aggregate and disaggregate belief updating is analogous. Prior beliefs regarding skill are

$$\theta = \overline{\theta} + \varepsilon_\theta$$

where

$$\varepsilon_\theta \sim N\left(0, \sigma_\theta^2\right).$$

Production evidence is

$$q = \alpha\theta + \beta e + \varepsilon$$

where

$$\varepsilon \sim N\left(0, \sigma_\varepsilon^2\right).$$

Taking α, β, and e as known constants, we have

$$q \sim N\left(\alpha\overline{\theta} + \beta e, \alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2\right).$$

Hence,

$$\left[\begin{array}{c} \theta \\ q \end{array}\right] \sim N\left(\left[\begin{array}{c} \overline{\theta} \\ \alpha\overline{\theta} + \beta e \end{array}\right], \left[\begin{array}{cc} \sigma_\theta^2 & \alpha\sigma_\theta^2 \\ \alpha\sigma_\theta^2 & \alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2 \end{array}\right]\right).$$

Then, posterior beliefs regarding skill given production evidence is

$$(\theta \mid q = q_0) \sim N\left(E\left[\theta \mid q = q_0\right], Var\left[\theta \mid q\right]\right)$$

where

$$\begin{aligned} E\left[\theta \mid q = q_0\right] &= \overline{\theta} + \frac{\alpha\sigma_\theta^2}{\alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2}\left(q_0 - \alpha\overline{\theta} - \beta e\right) \\ &= \frac{\overline{\theta}\sigma_\varepsilon^2 + \alpha\sigma_\theta^2\left(q_0 - \beta e\right)}{\alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2} \end{aligned}$$

and

$$\begin{aligned} Var\left[\theta \mid q\right] &= \sigma_\theta^2 - \frac{\left(\alpha\sigma_\theta^2\right)^2}{\alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2} \\ &= \frac{\sigma_\theta^2\sigma_\varepsilon^2}{\alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2}. \end{aligned}$$

Write prior and evidence precision as $\tau_\theta = \frac{1}{\sigma_\theta^2}$ and $\tau_\varepsilon = \frac{1}{\sigma_\varepsilon^2}$. Then, we can rewrite posterior parameters in terms of weighted precisions.

$$E\left[\theta \mid q = q_0\right] = \frac{\overline{\theta}\tau_\theta + \alpha\tau_\varepsilon\left(q_0 - \beta e\right)}{\tau_\theta + \alpha^2\tau_\varepsilon}$$

and

$$Var\left[\theta \mid q\right] = \frac{1}{\tau_\theta + \alpha^2\tau_\varepsilon}.$$