Ralph's Implicit Incentives[1]

Ralph, a risk neutral individual, manages two products. Ralph and others who value his services (the market, also risk neutral) understand he impacts production as follows.

$$\begin{aligned} q_1 &= \alpha_1\theta + \beta_1 e + \varepsilon_1 \\ q_2 &= \alpha_2\theta + \beta_2 e + \varepsilon_2 \end{aligned}$$

where q_i is production of product i, θ represents Ralph's permanent skill contribution, e represents Ralph's transitory effort, ε_i represents mean zero normally distributed variation in production (variance σ_ε^2) with precision $\tau_\varepsilon = \frac{1}{\sigma_\varepsilon^2}$, and α_i represents product i's sensitivity to skill while β_i represents product i's sensitivity to effort (for simplicity, α_i and β_i are each normalized to sum to one; $\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = 1$). Production is observable to all parties but not contractible as it is not mutually verifiable. Hence, Ralph does not have a pay-for-performance (explicit incentives) employment contract. Alternatively, Ralph is paid a market wage that reflects his perceived (permanent or skill) value.[2]

However, since production bundles skill and effort contributions, Ralph may try to finesse the market's perception of his skill by increasing effort. The market recognizes such posturing motives and conjectures effort $\widehat{e}$. Given this conjecture, evidence regarding Ralph's skill from observed production is

$$\Im(\theta) = \frac{q_i - \beta_i \widehat{e}}{\alpha_i}$$

Prior to first period production, Ralph's wage, w_1, is equal to the market's prior beliefs (with precision τ_θ) regarding Ralph's skill, $\overline{\theta}$. After production is reported, Ralph's wage, w_2, reflects revised beliefs regarding Ralph's skill. Importantly, production reports can be aggregated, $q_1 + q_2$, or disaggregated, (q_1, q_2). The market's updated expectation of Ralph's skill given aggregate production and conjectured effort, $\widehat{e}$, is

$$E[\theta \mid q_1 + q_2, \widehat{e}] = \frac{\tau_\theta \overline{\theta} + \tau_\varepsilon/2\,(q_1 + q_2 - \widehat{e})}{\tau_\theta + \tau_\varepsilon/2}$$

Similarly, the market's updated expectation of Ralph's skill given disaggregate production and conjectured effort, $\widehat{e}$, is

$$E[\theta \mid q_1, q_2, \widehat{e}] = \frac{\tau_\theta \overline{\theta} + \tau_\varepsilon \alpha_1^2 (q_1 - \beta_1 \widehat{e})/\alpha_1 + \tau_\varepsilon \alpha_2^2 (q_2 - \beta_2 \widehat{e})/\alpha_2}{\tau_\theta + \tau_\varepsilon \alpha_1^2 + \tau_\varepsilon \alpha_2^2}$$

Ralph's expected utility for discounted (at rate δ) expected continuation wages, w_2, based on aggregate production is

$$EU = \delta E[\theta \mid q_1 + q_2, \widehat{e}] - v(e)$$

[1] Adapted from Arya and Mittendorf, "The benefits of aggregate performance metrics in the presence of career concerns."

[2] Recall flat wages are ineffective in supplying incentives for transitory inputs like effort.

where $v(e) = \frac{1}{2}e^2$ represents Ralph's personal cost of effort. Ralph, of course, chooses e and the first order condition for the maximization problem is

$$\begin{aligned} \frac{\delta\tau_\varepsilon/2}{\tau_\theta + \tau_\varepsilon/2} &= \frac{dv(e)}{de} \\ \frac{\delta\tau_\varepsilon/2}{\tau_\theta + \tau_\varepsilon/2} &= e^A \end{aligned}$$

$e^A = \widehat{e}^A$ is the (rational expectations) equilibrium for aggregate production.

Likewise, Ralph's expected utility for discounted expected continuation wages, w_2, based on disaggregate production is

$$EU = \delta E[\theta \mid q_1, q_2, \widehat{e}] - v(e)$$

The first order condition for the disaggregate production maximization problem is

$$\begin{aligned} \frac{\delta\tau_\varepsilon(\alpha_1\beta_1 + \alpha_2\beta_2)}{\tau_\theta + \tau_\varepsilon\alpha_1^2 + \tau_\varepsilon\alpha_2^2} &= \frac{dv(e)}{de} \\ \frac{\delta\tau_\varepsilon(\alpha_1\beta_1 + \alpha_2\beta_2)}{\tau_\theta + \tau_\varepsilon\alpha_1^2 + \tau_\varepsilon\alpha_2^2} &= e^D \end{aligned}$$

$e^D = \widehat{e}^D$ is the (rational expectations) equilibrium for disaggregate production. The sequence of events is summarized in the following timeline.

Ralph is offered an in initial wage, w_1	Ralph exerts effort e	labor market learns either $x_1 + x_2$ or (x_1, x_2)	labor market determines Ralph's continuation wage, w_2

timeline

Suppose common knowledge parameters are

$$\begin{array}{ll} \alpha_1 = \frac{2}{3} & \alpha_2 = \frac{1}{3} \\ \beta_1 = \frac{1}{3} & \beta_2 = \frac{2}{3} \\ \bar{\theta} = \frac{1}{2} & \tau_\theta = \frac{1}{3} \\ \tau_\varepsilon = \frac{1}{2} & \\ \delta = 1 & \end{array}$$

While only Ralph knows his skill is $\theta = \frac{5}{11}$.[3]

[3] You might find it interesting to contrast your results with $\theta = \frac{6}{11}$ or $\theta = \bar{\theta}$.

Required:

1. For aggregate production, find Ralph's equilbrium effort level e^A, expected continuation wage $E\left[\theta \mid q_1 + q_2, \widehat{e}^A\right]$, and EU where $q_1 + q_2 = E\left[q_1 + q_2 \mid \theta\right]$.

2. For disaggregate production, find Ralph's equilbrium effort level e^D, expected continuation wage $E\left[\theta \mid q_1, q_2, \widehat{e}^D\right]$, and EU where $q_1 = E\left[q_1 \mid \theta\right]$ and $q_2 = E\left[q_2 \mid \theta\right]$.

3. It might be puzzling that the equilibrium involves Ralph supplying any effort since the market recognizes Ralph may posture via effort and Ralph dislikes effort. Verify that the effort strategies identified in 1 and 2 are equilibria by forming a normal form game with strategies $e = e^A$ and $e = 0$ for both Ralph and the market under aggregate production, and $e = e^D$ and $e = 0$ for both Ralph and the market under disaggregate production.
Expected utility for the market winner of Ralph's services is the expected benefit (say, $E\left[q_1 + q_2 \mid e, \widehat{e}\right]$) less w_2. You might find the following instructive (Ralph's expected utility is in the upper left of each cell).

		market	
		$e = \frac{3}{7}$	$e = 0$
Ralph	$e = \frac{3}{7}$	0.3887 0.4286	0.5724 0.2449
	$e = 0$	0.2968 0.1837	0.4805 0.0000

4. Expected surplus is $EU\left(Ralph\right) + EU\left(employer\right) = \left\{E\left[wage\right] - v\left(e\right)\right\} + \left\{E\left[q_1 + q_2\right] - E\left[wage\right]\right\} = E\left[q_1 + q_2\right] - v\left(e\right)$.[4] Is aggregate or disaggregate production more efficient (greater expected surplus)? For which features of this setting might accounting play a role?

[4] First-best marginal expected surplus in effort is

$$\begin{aligned} \frac{dE\left[q_1 + q_2\right]}{de} &= \frac{dv\left(e\right)}{de} \\ 1 &= e \end{aligned}$$

Appendix

Here we illustrate belief updating with one signal — aggregate and disaggregate belief updating is analogous. Prior beliefs regarding skill are

$$\theta = \overline{\theta} + \varepsilon_\theta$$

where

$$\varepsilon_\theta \sim N\left(0, \sigma_\theta^2\right).$$

Production evidence is

$$q = \alpha\theta + \beta e + \varepsilon$$

where

$$\varepsilon \sim N\left(0, \sigma_\varepsilon^2\right).$$

Taking α, β, and e as known constants, we have

$$q \sim N\left(\alpha\overline{\theta} + \beta e, \alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2\right).$$

Hence,

$$\left[\begin{array}{c} \theta \\ q \end{array}\right] \sim N\left(\left[\begin{array}{c} \overline{\theta} \\ \alpha\overline{\theta} + \beta e \end{array}\right], \left[\begin{array}{cc} \sigma_\theta^2 & \alpha\sigma_\theta^2 \\ \alpha\sigma_\theta^2 & \alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2 \end{array}\right]\right).$$

Then, posterior beliefs regarding skill given production evidence is

$$(\theta \mid q = q_0) \sim N\left(E\left[\theta \mid q = q_0\right], Var\left[\theta \mid q\right]\right)$$

where

$$\begin{aligned} E\left[\theta \mid q = q_0\right] &= \overline{\theta} + \frac{\alpha\sigma_\theta^2}{\alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2}\left(q_0 - \alpha\overline{\theta} - \beta e\right) \\ &= \frac{\overline{\theta}\sigma_\varepsilon^2 + \alpha\sigma_\theta^2\left(q_0 - \beta e\right)}{\alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2} \end{aligned}$$

and

$$\begin{aligned} Var\left[\theta \mid q\right] &= \sigma_\theta^2 - \frac{\left(\alpha\sigma_\theta^2\right)^2}{\alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2} \\ &= \frac{\sigma_\theta^2\sigma_\varepsilon^2}{\alpha^2\sigma_\theta^2 + \sigma_\varepsilon^2}. \end{aligned}$$

Write prior and evidence precision as $\tau_\theta = \frac{1}{\sigma_\theta^2}$ and $\tau_\varepsilon = \frac{1}{\sigma_\varepsilon^2}$. Then, we can rewrite posterior parameters in terms of weighted precisions.

$$E\left[\theta \mid q = q_0\right] = \frac{\overline{\theta}\tau_\theta + \alpha\tau_\varepsilon\left(q_0 - \beta e\right)}{\tau_\theta + \alpha^2\tau_\varepsilon}$$

and

$$Var\left[\theta \mid q\right] = \frac{1}{\tau_\theta + \alpha^2\tau_\varepsilon}.$$