Ralph's heterogeneous partnership[1]

Part A:

Alice and Ralph are contemplating a partnership venture in which each dollar invested returns an uncertain amount y. They can borrow any amount a and repay $ha = 1.02a$ such that the payoff is $x = ay - 1.02a$. Alice assigns y a normal distribution with mean $m_A = 50$ and variance $v = 1,000$ while Ralph assigns y a normal distribution with mean $m_R = 60$ and variance $v = 1,000$. Alice's preferences are represented by $U_A(x) = -\exp\left[-\frac{x}{\tau_A}\right] = -\exp[-0.008x]$ and Ralph's preferences are represented by $U_R(x) = -\exp\left[-\frac{x}{\tau_R}\right] = -\exp[-0.002x]$.

Suggested:

1. Suppose Alice and Ralph each pursue the investment opportunity on their own. Determine the optimal amount of borrowing a (in other words, the scale of the operation) and the corresponding certainty equivalent for each. (Hint: the certainty equivalent with exponential utility, linear payoffs with a normal distribution is $CE = E[x] - \frac{1}{2\tau}Var[x]$ where τ is risk tolerance or inverse of risk aversion $\frac{1}{\rho}$).

The following efficient risk sharing rules for the partnership are Pareto optimal or efficient.

$$\begin{aligned} s_A &= x\frac{\tau_A}{T} + \\ &\quad \left(\frac{1}{\tau_A} + \frac{1}{\tau_R}\right)^{-1} \frac{1}{2}\left[-\frac{(y - m_A)^2}{v} + \frac{(y - m_R)^2}{v} + 2\log\frac{\alpha_A}{1 - \alpha_A}\right] \\ &= x\frac{\tau_A}{T} + \\ &\quad \left(\frac{1}{\tau_A} + \frac{1}{\tau_R}\right)^{-1} \left[\frac{(y - (m_A + m_R)/2)(m_A - m_R)}{v} + \log\frac{\alpha_A}{1 - \alpha_A}\right] \end{aligned}$$

and

$$\begin{aligned} s_R &= x\frac{\tau_R}{T} + \\ &\quad \left(\frac{1}{\tau_A} + \frac{1}{\tau_R}\right)^{-1} \frac{1}{2}\left[-\frac{(y - m_R)^2}{v} + \frac{(y - m_A)^2}{v} + 2\log\frac{1 - \alpha_A}{\alpha_A}\right] \\ &= x\frac{\tau_R}{T} + \\ &\quad \left(\frac{1}{\tau_A} + \frac{1}{\tau_R}\right)^{-1} \left[\frac{(y - (m_R + m_A)/2)(m_R - m_A)}{v} + \log\frac{1 - \alpha_A}{\alpha_A}\right] \end{aligned}$$

[1] This example draws from Wilson, 1968, "The theory of syndicates," *Econometrica 36 no. 1*, 119-132.

where $T = \tau_A + \tau_R$, $s_A + s_R = x$, and α_A ($\alpha_R = 1 - \alpha_A$) is the weight on Alice's (Ralph's) expected utility to form the partnership's preference measure.

$$E_0 [U_0 (x)] = \alpha_A E_A [U_A (s_A x)] + (1 - \alpha_A) E_R [U_R (s_R x)]$$

Assigning the following parameters gives a consistent partnership normal probability distribution over y with variance mean $m_0 = v \left(\frac{\tau_A m_A}{T v_A} + \frac{\tau_R m_R}{T v_R} \right)$. Then, the partnership's certainty equivalent is

$$CE_0 = a m_0 - 1.02 a_0 - \frac{1}{2T} v a_0 h^2$$

and the partnership's optimal borrowing (scale of production) chooses a_0 to maximize CE.

$$a_0^* = T \frac{m_0 - 1.02}{v}$$

2. Find the partnership's optimal borrowing a_0^*.

3. If $\alpha_A = 0.36823$, determine Alice's and Ralph's certainty equivalents associated with forming the partnership and operating at a_0^*. (Hint: the sharing rules $s_A = b_A + d_A y$ and $s_R = b_R + d_R y$ result in linear sharing arrangements where $b_A + b_R = -1.02 a_0$ and $d_A + d_R = a_0$.) How does this compare with Alice and Ralph pursuing the project on their own? Compare $\frac{d_A}{a_0}$ with $\frac{a_A}{a_0}$ and $\frac{d_R}{a_0}$ with $\frac{a_R}{a_0}$.

4. Suppose only Alice or Ralph or collectively as partners can pursue the project (without substantially impairing its attractiveness), who should own it Alice, Ralph, or the partnership? (Hint: you might explore Alice's and Ralph's certainty equivalent operating at a_0^*.)

Part B:

Suppose beliefs and preferences remain as above except Ralph assigns variance of y equal to $v_R = 1,200$ (on the other hand, Alice continues to assign variance equal to $v_A = 1,000$). This added heterogeneity of beliefs results in modified Pareto efficient sharing rules.

$$\begin{aligned} s_A &= x\frac{\tau_A}{T} + \\ & \left(\frac{1}{\tau_A} + \frac{1}{\tau_R}\right)^{-1} \frac{1}{2} \left[-\frac{(y - m_A)^2}{v_A} + \frac{(y - m_R)^2}{v_R} - \log \frac{v_A}{v_R} + 2 \log \frac{\alpha_A}{1 - \alpha_A} \right] \end{aligned}$$

and

$$\begin{aligned} s_R &= x\frac{\tau_R}{T} + \\ & \left(\frac{1}{\tau_A} + \frac{1}{\tau_R}\right)^{-1} \frac{1}{2} \left[-\frac{(y - m_R)^2}{v_R} + \frac{(y - m_A)^2}{v_A} - \log \frac{v_R}{v_A} + 2 \log \frac{1 - \alpha_A}{\alpha_A} \right] \end{aligned}$$

As the sharing rules produce quadratic rather than linear payments in y, the partner's certainty equivalent is modified (on the other hand, operating individually the payoffs continue to be linear in y). Let a partner's payments be

$$c_0^j + c_1^j y + c_2^j y^2, \quad j = A, R$$

then the partner's certainty equivalent is

$$\begin{aligned} CE_j &= \tau_j \left\{ \frac{v_j \left(4c_0c_2 - c_1^2\right) + 2\tau_j \left(c_0 + c_1 m_j + c_2 m_j^2\right)}{2\tau_j \left(2c_2 v_j + \tau_j\right)} + \frac{1}{2} \log \left(1 + \frac{2c_2 v_j}{\tau_j}\right) \right\} \\ \text{for } j &= A, R \end{aligned}$$

and the partnership's probability measure has variance

$$v_0 = \left(\frac{\tau_A}{Tv_A} + \frac{\tau_R}{Tv_R}\right)^{-1}$$

and mean

$$m_0 = v_0 \left(\frac{\tau_A m_A}{Tv_A} + \frac{\tau_R m_R}{Tv_R}\right)$$

Expanding and simplifying the sharing rules gives

$$\begin{aligned} s_A &= c_0^A + c_1^A y + c_2^A y^2 \\ &= \frac{\tau_A}{2Tv_Av_R} \times \\ & \left[\tau_R \left(m_R^2 v_A - m_A^2 v_R\right) - 2a_0 h v_A v_R + \tau_R v_A v_R \left(2 \log \frac{\alpha_A}{\alpha_R} - \log \frac{v_A}{v_R}\right) \right] \\ & + \tau_A \frac{\tau_R \left(m_A v_R - m_R v_A\right) + a_0 v_A v}{Tv_Av_R} y \\ & + \frac{\tau_A \tau_R \left(v_A - v_R\right)}{2Tv_Av_R} y^2 \end{aligned}$$

and

$$
\begin{aligned}
s_R &= c_0^R + c_1^R y + c_2^R y^2 \\
&= \frac{\tau_R}{2T v_A v_R} \times \\
&\quad \left[\tau_A \left(m_A^2 v_R - m_R^2 v_A\right) - 2a_0 h v_A v_R + \tau_A v_A v_R \left(2 \log \frac{\alpha_R}{\alpha_A} - \log \frac{v_R}{v_A}\right)\right] \\
&\quad + \tau_R \frac{\tau_A \left(m_R v_A - m_A v_R\right) + a_0 v_A v_R}{T v_A v_R} y \\
&\quad + \frac{\tau_A \tau_R \left(v_R - v_A\right)}{2T v_A v_R} y^2
\end{aligned}
$$

where $\alpha_R = 1 - \alpha_A$.

Suggested:

1. Suppose Alice and Ralph each pursue the investment opportunity on their own. Determine the optimal amount of borrowing a (in other words, the scale of the operation) and the corresponding certainty equivalent for each.

2. Find the partnership's optimal borrowing a_0^*.

3. If $\alpha_A = 0.437421$, determine Alice's and Ralph's certainty equivalents associated with forming the partnership and operating at a_0^*.

4. Should Alice, Ralph, or the partnership own the project? Explain.