Ralph's equilibrium probability assignment

Ralph recognizes there is an abundance of information in equilibrium prices. Ralph frames asset dynamics as a Markov process where payoffs depend on transitions from one to another state with the immediate previous state a sufficient description of the asset's history. For instance, a two-state economy involves

$$A_1 = \begin{array}{rc} & \begin{array}{cc} s_{11} & s_{12} \end{array} \\ \begin{array}{r} asset_1 \\ asset_2 \end{array} & \begin{bmatrix} 1.05 & 1.05 \\ 2 & 0 \end{bmatrix} \end{array}, \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$A_2 = \begin{array}{rc} & \begin{array}{cc} s_{21} & s_{22} \end{array} \\ \begin{array}{r} asset_1 \\ asset_2 \end{array} & \begin{bmatrix} 1.05 & 1.05 \\ 0 & 3 \end{bmatrix} \end{array}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where A_i reflects payoffs associated with transitioning from initial state i to state j, s_{ij}, and x_i represents nominal prices or investment cost of the assets. Arrow-Debreu or state prices (the equilibrium price of an asset that pays in only one state and zero otherwise) are the solution, $y_i > 0$, to

$$A_i y_i = x_i$$

Ross' recovery theorem. Ralph knows Ross' recovery theorem assigns state-transition probability beliefs, F, for a representative investor based on equilibrium state prices, P.

$$F = \frac{1}{\delta} D P D^{-1}$$

where element F_{ij} is the transition probability from state i to state j, element P_{ij} is the state price associated with transitioning from state i to state j, δ is a representative investor's personal discount factor, and D is a diagonal matrix with element D_{jj} depicting a representative investor's marginal utility for consumption in state j. Ralph knows $D = I$ when the riskless rate of return is state independent (the same in all initial states) and δ equals the reciprocal of one plus the riskless rate of return. Otherwise, δ and D are derived from eigensystem decomposition of P.

$$Pz = \delta z$$

where δ is the largest eigenvalue of P and z is its associated eigenvector which is the main diagonal of D^{-1}. As eigenvectors are scale-free, all that matters is that all elements of z have the same sign (can be made positive) and their relative magnitudes then describe marginal rates of substitution for consumption between states.

Kelly criterion. Ralph knows an investor who seeks to maximize long-run wealth employs the Kelly criterion. For each initial state i this involves portfolio weights on Arrow-Debreu assets, ν_i, equal to the (transition) probability assigned to the state, $\nu_i = F_i^T$, where F_i is row i from F.

$$\begin{aligned} E[r_i] &= F_i \log(\Omega_i v_i) \\ &= F_i \log\left(A_i^T w_i\right) \end{aligned}$$

where $\Omega_i = \begin{bmatrix} \frac{1}{y_{i1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{y_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \frac{1}{y_{in}} \end{bmatrix}$ and optimal weights on nominal assets, w_i, are

$$w_i = \left(A_i^T\right)^{-1} \Omega_i F_i^T$$

Suggested:

1. Ross' recovery theorem.

a. Find state prices for A_1 and A_2 and create P.

b. Find the largest eigenvalue of P, δ, and its associated eigenvector, z. Hint: Is the average of the sum of the rows of P the largest eigenvalue, δ — beware, this is a highly specialized case where every row of P has the same sum? The associated eigenvector is a vector in the nullspace of $(P - \delta I)$.

c. Assign state-transition probability beliefs, F, for a representative investor.

d. Find steady-state probabilities (state probabilities to which the dynamic system converges) such that $p_{ss}^T F = p_{ss}^T$.

2. Kelly criterion. Suppose a representative investor assigns state-transition probabilities based on Ross' recovery theorem (as above) but employs a Kelly investment strategy (maximizes expected long-run wealth).

a. Find the expected periodic logarithmic return for a representative Kelly-strategy investor in each initial state $E[r_i]$.

b. Find the expected periodic asset growth in each initial state $G[r_i] = \exp(E[r_i])$.

c. What are the optimal weights on the nominal assets, w_i?

d. Find $E[r] = p_{ss}^T E[r_i]$ and $G[r] = \exp(E[r])$. How does $G[r]$ compare with $\frac{1}{\delta}$?

Example 2.

$$A_1 = \begin{array}{cc} & \begin{array}{cc} s_{11} & s_{12} \end{array} \\ \begin{array}{c} asset_1 \\ asset_2 \end{array} & \begin{bmatrix} 1.75 & 0 \\ 0 & 2.625 \end{bmatrix} \end{array}, \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$A_2 = \begin{array}{cc} & \begin{array}{cc} s_{21} & s_{22} \end{array} \\ \begin{array}{c} asset_1 \\ asset_2 \end{array} & \begin{bmatrix} 2.625 & 0 \\ 0 & 1.75 \end{bmatrix} \end{array}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Suggested: 3. repeat 1 and 2 for example 2.

Example 3.

$$A_1 = \begin{array}{cc} & \begin{array}{ccc} s_{11} & s_{12} & s_{13} \end{array} \\ \begin{array}{c} asset_1 \\ asset_2 \\ asset_3 \end{array} & \begin{bmatrix} 1.00 & 1.00 & 1.00 \\ 3.00 & 0 & 0 \\ 0 & 2.00 & 1.00 \end{bmatrix} \end{array}, \quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{array}{cc} & \begin{array}{ccc} s_{21} & s_{22} & s_{23} \end{array} \\ \begin{array}{c} asset_1 \\ asset_2 \\ asset_3 \end{array} & \begin{bmatrix} 1.05 & 1.05 & 1.05 \\ 3.15 & 0 & 0 \\ 0 & 2.10 & 1.05 \end{bmatrix} \end{array}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$A_3 = \begin{array}{cc} & \begin{array}{ccc} s_{31} & s_{32} & s_{33} \end{array} \\ \begin{array}{c} asset_1 \\ asset_2 \\ asset_3 \end{array} & \begin{bmatrix} 1.10 & 1.10 & 1.10 \\ 3.30 & 0 & 0 \\ 0 & 2.20 & 1.10 \end{bmatrix} \end{array}, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Suggested: 4. repeat 1 and 2 for example 3.

Example 4.

$$A_1 = \begin{array}{cc} & \begin{array}{ccc} s_{11} & s_{12} & s_{13} \end{array} \\ \begin{array}{c} asset_1 \\ asset_2 \\ asset_3 \end{array} & \begin{bmatrix} 2.00 & 1.00 & 0 \\ 1.00 & 0 & 2.00 \\ 0 & 2.00 & 1.00 \end{bmatrix} \end{array}, \quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{array}{cc} & \begin{array}{ccc} s_{21} & s_{22} & s_{23} \end{array} \\ \begin{array}{c} asset_1 \\ asset_2 \\ asset_3 \end{array} & \begin{bmatrix} 2.10 & 1.05 & 0 \\ 1.05 & 0 & 2.10 \\ 0 & 2.10 & 1.05 \end{bmatrix} \end{array}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$A_3 = \begin{array}{c} \\ asset_1 \\ asset_2 \\ asset_3 \end{array} \begin{bmatrix} s_{31} & s_{32} & s_{33} \\ 2.20 & 1.10 & 0 \\ 1.10 & 0 & 2.20 \\ 0 & 2.20 & 1.10 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Suggested: 5. repeat 1 and 2 for example 4.

Appendix

An iterative algorithm for finding the largest eigenvalue (in absolute value) for a square matrix, A.

1. Let k_1 be any vector, say, a vector of ones (scaled to unit length) where the number of elements in the vector equals the number of rows or columns in A.

2. Let $k_{t+1} = \frac{Ak_t}{\mu}$ where $\mu = \frac{k_t^T A k_t}{k_t^T k_t}$. Normalize k_{t+1} before proceeding with the next iteration, that is, set $k_{t+1} = \frac{k_{t+1}}{\sqrt{k_{t+1}^T k_{t+1}}}$.

3. iterate until $k_{t+1} = k_t$ (actually sufficiently close, say, less than 10^{-10}).

4. μ is the largest eigenvalue of A and $k_t = k_{t+1}$ is it's associated eigenvector.