Ralph's Binomial Sanitization[1]

As board chairman, Alice is interested in hiring a highly-skilled manager, Ralph, to operate her multi-project firm. Alice recognizes the manager's skill (and actions) impacts the likelihood of project success (r) as follows.

manager's skill	r
H	$\frac{2}{3}$
L	$\frac{1}{3}$

Each success produces a (compound) return of $u = 1.10$ while failure produces a return of $d = \frac{1}{1.10}$. For simplicity, the number of projects is $n = 2$ and r is independent across projects. Hence, it is common knowledge (from the binomial distribution; $\Pr(s, n) = \begin{pmatrix} n \\ s \end{pmatrix} r^s (1-r)^{n-s}$) the likelihood of s successes out of n projects and the return from the projects at time 2 are as tabulated below.

(s, f)	$(0, 2)$	$(1, 1)$	$(2, 0)$	$E[\text{return} \mid \text{skill}]$
return	$d^2 = \frac{1}{1.21}$	$ud = du = 1$	$u^2 = 1.21$	$[r_j u + (1 - r_j) d]^2$
$\Pr(s, f \mid j = H)$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$	1.0740
$\Pr(s, f \mid j = L)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	0.9462

Ex post (time 2) returns and likelihoods

Projects are completed over two periods but the manager's talents are especially vital to first period operations. Alice believes Ralph is skilled but as this is privately known to Ralph, Alice elects to offer compensation (cash and ownership) that pays a normalized amount equal to the return to the firm at time 1. Alice recognizes that managers outside employment opportunities pay a (normalized) certainty equivalent equal to 1 (hence, a risk neutral manager would only accept Alice's employment if of skill H since $1.0740 > 1 > 0.9462$). Ralph is risk averse with utility for wealth w

$$U(w) = -\exp[-w]$$

where Ralph's compensation at time 1 depends on his audited productivity report. Ralph is compensated and leaves the firm at time 1. At time 2, project

[1] This example is drawn from ideas developed in Shin, 2003, "Disclosures and asset returns," *Econometrica*, 71(1), 105-133.

success is known and the firm is liquidated. The timeline is below.

time 0	time 1	time 2
the firm is organized and Ralph is hired	Ralph acquires private information, provides audited report, is compensated, and leaves the firm	firm is liquidated and owners receive their distributions

timeline

Except for the auditing stipulation, Alice leaves the time 1 reporting choice to Ralph. Ralph can announce the disclosure strategy and faithfully provide full disclosure of project successes and failures (*full*), faithfully provide a sanitized report of successes with failures suppressed (*san*), or faithfully provide a conservative report with failures disclosed and successes suppressed (*con*). Ralph privately learns at time 1 with probability $\theta = \frac{1}{3}$ the fate of each project (θ is independent across projects).

Since Ralph's compensation is pegged to period 1 firm value, Ralph evaluates his disclosure options based on time 1 equilibrium firm value conditional on his disclosure strategy (V_1^y where $y \in \{full, san, con\}$). Time 1 equilibrium value equals the expected payoff conditional on Ralph's strategic report,

$$\begin{aligned} V_1^y &= u^s d^f \sum_{x=0}^{n-s-f} \binom{n-s-f}{x} (ur)^x \left(d\left(1-r\right)\right)^{n-s-f-x} \\ &= u^s d^f \left[ru + \left(1-r\right) d\right]^{n-s-f} \end{aligned}$$

where s is reported successes, f is reported failures, n is the number of projects, and r is the probability of project success conditional on the manager's type.

$(s, f \mid full)$	$\Pr(s, f \mid full)$	$V_1^{full}(s, f)$
$(0, 0)$	$(1-\theta)^2$	$(ru + (1-r)\,d)^2$
$(0, 1)$	$2\theta(1-\theta)(1-r)$	$d\,(ru + (1-r)\,d)$
$(0, 2)$	$\theta^2(1-r)^2$	d^2
$(1, 0)$	$2\theta(1-\theta)\,r$	$u\,(ru + (1-r)\,d)$
$(1, 1)$	$2\theta^2 r(1-r)$	ud
$(2, 0)$	$\theta^2 r^2$	u^2

Time 1 reports, likelihoods, and equilibrium firm value for full disclosure strategy[2]

$(s, f \mid san)$	$\Pr(s, f \mid san)$	$V_1^{san}(s, f)$
$(0, 0)$	$(1-r\theta)^2$	$\frac{(ru(1-\theta)+(1-r)d)^2}{(1-r\theta)^2}$
$(1, 0)$	$2\theta(1-r\theta)\,r$	$\frac{u(ru(1-\theta)+(1-r)d)}{1-r\theta}$
$(2, 0)$	$\theta^2 r^2$	u^2
$(0, 1)^*$	NA	$d\,(ru + (1-r)\,d)$
$(0, 2)^*$	NA	d^2
$(1, 1)^*$	NA	ud

Time 1 reports, likelihoods, and equilibrium firm value for sanitization strategy
*represents an off-equilibrium report by the manager

[2] When only one signal is realized at time 1, there are two ways for realization to occur — the first project or the second project.

$(s,f \mid con)$	$\Pr(s,f \mid con)$	$V_1^{con}(s,f)$
$(0,0)$	$[1-(1-r)\theta]^2$	$\frac{(ru+(1-\theta)(1-r)d)^2}{[1-(1-r)\theta]^2}$
$(0,1)$	$2\theta(1-r)[1-(1-r)\theta]$	$\frac{d(ru+(1-\theta)(1-r)d)}{1-(1-r)\theta}$
$(0,2)$	$\theta^2(1-r)^2$	d^2
$(1,0)^*$	NA	$u(ru+(1-r)d)$
$(1,1)^*$	NA	ud
$(2,0)^*$	NA	u^2

Time 1 reports, likelihoods, and equilibrium firm value
for conservative strategy
*represents an off-equilibrium report by the manager

Off-equilibrium reports are not expected but arise as a result of reporting errors, the so-called "trembling hand." As the audited results are faithful, full disclosure equilibrium prices follow. However, it could be argued the auditor would eliminate reports which are inconsistent with "declared" report strategy. Hence, negligible likelihood is associated with off-equilibrium reports.

Part A

Required:

1. Determine Ralph's certainty equivalent if he employs the full disclosure strategy and is type H or if he is type L. Will Ralph accept Alice's employment if he is type H? If he is type L? Does type H first order stochastic dominate type L?[3]

2. Repeat 1 for the sanitization report strategy.

3. Repeat 1 for the conservative report strategy.

4. Which report strategy will Ralph choose, full disclosure, sanitization, or conservative? Are any of these report strategies a mean preserving spread of other strategies (that is, are the means equal and does second order stochastic dominance apply to any pairs of gambles)?

5. Suppose Alice is not committed to audited performance reporting and the firm's shares are priced at their equilibrium value, does Ralph prefer a sanitization reporting strategy or no time 1 report? What does this suggest about the relative importance of valuation versus evaluation for accounting? In other words, is the role of accounting more substantive in a world of pure exchange or a world of production?

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[3] You may want to review the stochastic dominance and mean preserving spread discussion in Ralph's Sanitization part B.

Part B[4]

Suppose the number of projects, n, is very large (unbounded, $n \to \infty$, $r \to 0$, and $rn \to \lambda$). In this setting, the binomial distribution becomes cumbersome but with a little thought Ralph recognizes that the binomial distribution approaches the Poisson distribution when n becomes very large. The probability mass function for a Poisson random variable s is

$$\Pr(s) = \frac{e^{-\lambda}\lambda^s}{s!}, \quad \lambda > 0, s \geq 0$$

where s represents number of successes in a fixed interval, and λ is both the expected value of s and the variance of s. To see this connection, first Ralph recognizes the expected value of a binomial random variable is $n*r$. Equating $n*r$ and λ leads to $r = \frac{\lambda}{n}$.[5] Next, Ralph recognizes the large sample approximation of $\binom{n}{s} = \frac{n(n-1)\cdots(n-s+1)}{s!}$ can be thought of as $\frac{n^s}{s!}$ (since all s terms in the numerator are extremely large they can be considered identical). Now, Ralph substitutes these results into the binomial mass function

$$\begin{aligned} \Pr(s,n) &= \begin{pmatrix} n \\ s \end{pmatrix} r^s (1-r)^{n-s} \\ &= \frac{n^s}{s!}\left(\frac{\lambda}{n}\right)^s \left(1-\frac{\lambda}{n}\right)^{n-s} \end{aligned}$$

From the idea of continuous compounding (as s is small relative to n, s can be ignored), Ralph recognizes the limit of the last term, $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-s} = e^{-\lambda}$. Some rearrangement of the first two terms leads to $\frac{1}{s!}\left(\frac{n\lambda}{n}\right)^s = \frac{\lambda^s}{s!}$, a term that doesn't involve n. Hence, Ralph now has

$$\lim_{n\to\infty} \Pr(s,n) = \Pr(s) = e^{-\lambda}\frac{\lambda^s}{s!}$$

completing Ralph's connection of the binomial distribution to the Poisson distribution when the number of trials (projects) is very large.

To keep things interesting, let $d = 1$, $\lambda_H = \frac{4}{3}$, and $\lambda_L = \frac{2}{3}$. In other words, failed projects leave firm value unchanged.[6] The expected return conditional on skill now becomes

$$\begin{aligned} \lim_{n\to\infty}\left[r_j u + (1-r_j)\, d\right]^n &= \lim_{n\to\infty}\left[1 + r_j (u-1)\right]^n \\ &= \lim_{n\to\infty}\left[1 + \frac{\lambda_j}{n}(u-1)\right]^n \\ &= \exp\left[\lambda_j (u-1)\right] \end{aligned}$$

[4] This part draws heavily upon Shin, "Disclosure risk and price drift," *Journal of Accounting Research*, May 2006, 351-379.

[5] The variance of a binomial random variable is $nr(1-r)$. As $r \to 0$ and $nr \to \lambda$, Ralph recognizes $nr(1-r) \to nr = \lambda$ which implies the Poisson limit of a binomial has the same expected value and variance — the properties of a Poisson random variable.

[6] For $0 < d < 1$, $r < 1$, and $n \to \infty$, $d^n \to 0$ — an absorbing state.

Hence, $E[return \mid H] = 1.1426$ and $E[return \mid L] = 1.0689$. Ralph's reservation wage in this setting is 1.1 (normalized).

Ralph focuses on only the sanitization disclosure strategy. The probability of $s+k$ realized successes at the terminal date (time 2) given s disclosed successes at interim date (time 1) is $e^{-\lambda(1-\theta)} \frac{[\lambda(1-\theta)]^k}{k!}$. Ralph recognizes this is a Poisson distribution with characteristic parameter $\lambda(1-\theta)$ (also its mean and variance). Therefore, the expected liquidation value following s disclosed successes at time 1 is

$$\begin{aligned} V_1(s,j) &= u^s \sum_{k=0}^{n} e^{-\lambda_j(1-\theta)} \frac{[\lambda_j(1-\theta)u]^k}{k!} \\ &= u^s \exp[\lambda_j(1-\theta)(u-1)] \end{aligned}$$

Required:

1. Compare the time 1 expected liquidation value (equal to Ralph's normalized compensation) for a high (H) versus low (L) with $s = 0$ to 10 successes reported at time 1 along with the probability of s successes reported at time 1, $\Pr(s) = e^{-\lambda_j \theta} \frac{[\lambda_j \theta]^s}{s!}$. Ralph believes the likelihood $s > 10$ is negligible.

2. Determine Ralph's certainty equivalent if he employs the sanitization disclosure strategy and is type H or if he is type L. Will Ralph accept Alice's employment if he is type H? If he is type L?