Ralph's Bayesian Accruals

Continue with the setting in Ralph's Optimal Accruals but build a frame beginning with m_0.

$$m_0 = \mu + \varepsilon_0, \quad \varepsilon_0 \sim N\left(0, a\sigma^2\right)$$

Then, cash flow information relates to m_0 and the quantity of interest, m_3, as follows.

$$\begin{aligned} y_1 &= \mu + \varepsilon_0 + \varepsilon_1 + e_1 \\ y_2 &= \mu + \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + e_2 \\ y_3 &= \mu + \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + e_3 \\ m_3 &= \mu + \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \end{aligned}$$

where $\eta^T = \begin{bmatrix} \varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & e_1 & e_2 & e_3 \end{bmatrix} \sim N(0, \Sigma)$, $\Sigma = \begin{bmatrix} a\sigma^2 & 0 \\ 0 & \sigma^2 I_6 \end{bmatrix}$. This can be compactly written

$$Y = \mu\iota + A\eta$$

where $Y^T = \begin{bmatrix} y_1 & y_2 & y_3 & m_3 \end{bmatrix}$, ι is a vector of ones, and

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Let $\sigma^2 = 1$. The joint distribution is

$$Y \sim N\left(\mu\iota, A\Sigma A^T\right)$$

Suggested:

1. Utilize the Bayes' normal theorem to derive the posterior (updated) distribution for m_3 conditional on y_1,y_2, and y_3 simultaneously. (Hint: m_3 continues to have a normal distribution but with potentially revised mean and variance.)
2. Utilize the Bayes' normal theorem to derive the posterior distribution (mean and variance) for y_2, y_3, and m_3 conditional on y_1.
3. Utilize the result in 2 to derive the posterior distribution for y_3 and m_3 conditional on y_1 and y_2.
4. Utilize the result in 3 to derive the posterior distribution for m_3 conditional on y_1, y_2, and y_3. How does this compare with 1?
5. Suppose m_0 is known, then $a = 0$. Evaluate your expression in 1 or 4.
6. Suppose m_0 is equally uncertain as m_t for $t > 0$, $a = 1$. Evaluate your expression in 1 or 4.
7. Suppose m_0 is extremely uncertain so as to be uninformative, $a \to \infty$. Evaluate your expression in 1 or 4.