1.2.1 *Probability as logic example*

Suppose we only know a variable, call it X_1, has support from $(-1, 1)$ and a second variable, X_2, has support from $(-2, 2)$. Then, we receive an aggregate report — their sum, $Y = X_1 + X_2$, equals $\frac{1}{2}$. What do we know about X_1 and X_2? Jayne's maximum entropy principle (*MEP*) suggests we assign probabilities based on what we know but only what we know.

Consider X_1 alone. Since we only know support, consistent probability assignment leads to the uniform density

$$f\left(X_1 : \{-1 < X_1 < 1\}\right) = \frac{1}{2}$$

Similarly,

$$f\left(X_2 : -2 < X_2 < 2\right) = \frac{1}{4}$$

and considered jointly we have[7]

$$f\left(X_1, X_2 : \{-1 < X_1 < 1, -2 < X_2 < 2\}\right) = \frac{1}{8}$$

Now, what is learned from the aggregate report $y = \frac{1}{2}$? Bayesian updating based on the evidence suggests

$$f\left(X_1 \mid y = \frac{1}{2}\right) = \frac{f\left(X_1, y = \frac{1}{2}\right)}{f\left(y = \frac{1}{2}\right)}$$

and

$$f\left(X_2 \mid y = \frac{1}{2}\right) = \frac{f\left(X_2, y = \frac{1}{2}\right)}{f\left(y = \frac{1}{2}\right)}$$

Hence, updating follows from probability assignment of $f\left(X_1, Y\right)$, $f\left(X_2, Y\right)$, and $f\left(Y\right)$. Since we have $f\left(X_1, X_2\right)$ and $Y = X_1 + X_2$ plus knowledge of any two of (Y, X_1, X_2) supplies the third, we know

$$f\left(X_1, Y : \left\{\begin{array}{c} \{-3 < Y < -1, -1 < X_1 < Y + 2\} \\ \{-1 < Y < 1, -1 < X_1 < 1\} \\ \{1 < Y < 3, Y - 2 < X_1 < 1\} \end{array}\right\}\right) = \frac{1}{8}$$

and

$$f\left(X_2, Y : \left\{\begin{array}{c} \{-3 < Y < -1, -2 < X_2 < Y + 1\} \\ \{-1 < Y < 1, -1 < X_2 < 1\} \\ \{1 < Y < 3, Y - 1 < X_2 < 2\} \end{array}\right\}\right) = \frac{1}{8}$$

Further,

$$\begin{aligned} f\left(Y\right) &= \int f\left(X_1, Y\right) dX_1 \\ &= \int f\left(X_2, Y\right) dX_2 \end{aligned}$$

Hence, integrating out X_1 or X_2 yields

$$\int_{-1}^{Y+2} f\left(X_1, Y\right) dX_1 = \int_{-1}^{Y+1} f\left(X_2, Y\right) dX_2$$

[7] *MEP* treats X_1 and X_2 as independent random variables as we have no knowledge regarding their relationship.

for $-3 < Y < -1$

$$\int_{-1}^{1} f(X_1, Y)\, dX_1 = \int_{-1}^{1} f(X_2, Y)\, dX_2$$

for $-1 < Y < 1$, and

$$\int_{Y-2}^{1} f(X_1, Y)\, dX_1 = \int_{Y-1}^{1} f(X_2, Y)\, dX_2$$

for $1 < Y < 3$. Collectively, we have[8]

$$\begin{aligned} f(Y : \{-3 < Y < -1\}) &= \frac{3+Y}{8} \\ f(Y : \{-1 < Y < 1\}) &= \frac{1}{4} \\ f(Y : \{1 < Y < 3\}) &= \frac{3-Y}{8} \end{aligned}$$

Now, conditional probability assignment given $y = \frac{1}{2}$ is

$$f\left(X_1 : \{-1 < X_1 < 1\} \mid y = \frac{1}{2}\right) = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$$

and

$$f\left(X_2 : \{Y - 1 < X_2 < Y + 1\} \mid y = \frac{1}{2}\right) = \frac{\frac{1}{8}}{\frac{1}{4}}$$

or

$$f\left(X_2 : \left\{-\frac{1}{2} < X_2 < \frac{3}{2}\right\} \mid y\right) = \frac{1}{2}$$

Hence, the aggregate report tells us nothing about X_1 (our unconditional beliefs are unaltered) but a good deal about X_2 (support is cut in half).

[8] Likewise, the marginal densities for X_1 and X_2 are identified by integrating out the other variable from their joint density. That is

$$\begin{aligned} & \int_{-2}^{2} f(X_1, X_2)\, dX_2 \\ = \quad & f(X_1 : \{-1 < X_1 < 1\}) = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} & \int_{-1}^{1} f(X_1, X_2)\, dX_1 \\ = \quad & f(X_2 : \{-2 < X_2 < 2\}) = \frac{1}{4} \end{aligned}$$

This consistency check brings us back to our starting point.

For instance, updated beliefs conditional on the aggregate report imply $E\left[X_1 \mid y = \frac{1}{2}\right] = 0$ and $E\left[X_2 \mid y = \frac{1}{2}\right] = \frac{1}{2}$. This is logically consistent as $E\left[X_1 + X_2 \mid y = \frac{1}{2}\right] = E\left[Y \mid y = \frac{1}{2}\right]$ must be equal to $\frac{1}{2}$.

On the other hand, if the aggregate report is $y = 2$, then revised beliefs are

$$f\left(X_1 : \{Y - 2 < X_1 < 1\} \mid y = 2\right) = \frac{\frac{1}{8}}{\frac{3-Y}{8}} = \frac{1}{3-2}$$

or

$$f\left(X_1 : \{0 < X_1 < 1\} \mid y = 2\right) = 1$$

and

$$f\left(X_2 : \{Y - 1 < X_2 < Y + 1\} \mid y = 2\right) = \frac{1}{3-2}$$

or

$$f\left(X_2 : \{1 < X_2 < 2\} \mid y = 2\right) = 1$$

The aggregate report is informative for both variables, X_1 and X_2. For example, updated beliefs imply

$$E\left[X_1 \mid y = 2\right] = \frac{1}{2}$$

and

$$E\left[X_2 \mid y = 2\right] = \frac{3}{2}$$

and

$$E\left[X_1 + X_2 \mid y = 2\right] = 2$$

Following a brief overview of chapter organization, we explore probability as logic in other accounting settings.