LEN model

The *LEN* model is a performance evaluation frame for dealing with unbounded performance measures. In particular, *LEN* stands for Linear compensation, negative Exponential utility, and Normally distributed performance measures. In standard incentive compensation fashion, the principal (say, owner) utilizes compensation to overcome the agent's (say, manager's) preference for input a_L as the principal prefers the manager supply a_H. The agent's equilibrium response follows from selecting input supply to maximize the agent's expected utility.

1 Agent's equilibrium behavior

Suppose compensation is linear (actually affine) in the performance measures, y.

$$I = \delta + \gamma^T y$$

where δ and γ are parameters chosen by the principal to align the agent's interest with the principals. The k performance measures are jointly normally distributed with mean μ_j, variance-covariance matrix Σ_j, and density function

$$f(y \mid a_j) = \frac{1}{(2\pi)^{k/2}|\Sigma_j|} \exp\left[-\frac{1}{2}\left(y-\mu_j\right)^T \Sigma_j^{-1}\left(y-\mu_j\right)\right], \quad j = L, H$$

And, the agent is risk averse with utility function for wealth w and personal cost of input $c(a)$, $a \in \{a_L, a_H\}$

$$U(w,a) = -\exp\left[-\rho\left(w - c(a)\right)\right]$$

The agent's compensation varies with the performance measures, y, which are normally distributed. Since compensation, I, is a linear combination of the performance measures, $\delta + \gamma^T y$, wealth also has a normal distribution with mean $m_j \equiv w_0 + \delta + \gamma^T \mu_j$ and variance $\sigma_j^2 \equiv \gamma^T \Sigma_j \gamma$, $j = L, H$. Hence, the density function for wealth is

$$f(w \mid a_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{\left(w - m_j\right)^2}{2\sigma_j^2}\right]$$

where wealth is initial wealth, w_0, plus current compensation, I

$$\begin{aligned} w &= w_0 + I \\ &= w_0 + \delta + \gamma^T y \end{aligned}$$

Then, the agent's expected utility is

$$E\left[U(w,a)\right] = \int_{-\infty}^{\infty} U(w,a)\, f(w \mid a_j)\, dw$$

$$
\begin{aligned}
E\left[U\left(w,a_j\right)\right] &= \int_{-\infty}^{\infty} U\left(w,a\right) f\left(w \mid a_j\right) dw \\
&= \int_{-\infty}^{\infty} -\exp\left[-\rho\left(w - c\left(a_j\right)\right)\right] f\left(w \mid a_j\right) dw \\
&= \int_{-\infty}^{\infty} -\exp\left[-\rho\left(w - c\left(a_j\right)\right)\right] \\
&\quad \times \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{\left(w - m_j\right)^2}{2\sigma_j^2}\right] dw \\
&= -\frac{1}{\sqrt{2\pi}\sigma_j} \int_{-\infty}^{\infty} \exp\left[-\rho\left(w - c\left(a_j\right)\right) - \frac{\left(w - m_j\right)^2}{2\sigma_j^2}\right] dw
\end{aligned}
$$

Now, the short version is to write the agent's expected utility as the utility of a certainty equivalent times the integral of a normal density and since the integral of a density function is one, we're left with only the utility of the certainty equivalent. This allows a very simple expression, the certainty equivalent, to identify the agent's equilibrium strategy for selecting input a.

Details are provided below. Rewrite the expected utility by collecting terms involving w.

$$
\begin{aligned}
E\left[U\left(w,a_j\right)\right] &= -\frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[\rho c\left(a_j\right)\right] \int_{-\infty}^{\infty} \exp\left[-\rho w - \frac{\left(w - m_j\right)^2}{2\sigma_j^2}\right] dw \\
&= -\frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[\rho c\left(a_j\right)\right] \int_{-\infty}^{\infty} \exp\left[-\frac{2\sigma_j^2 \rho w + \left(w - m_j\right)^2}{2\sigma_j^2}\right] dw
\end{aligned}
$$

Completing the square of the second exponential term gives

$$
\begin{aligned}
E\left[U\left(w,a_j\right)\right] &= -\frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[\rho c\left(a_j\right)\right] \\
&\quad \times \int_{-\infty}^{\infty} \exp\left[-\frac{2\sigma_j^2 \rho \left(m_j - \frac{\rho}{2}\sigma^2\right) + \left(w - m_j + \frac{\rho}{2}\sigma^2\right)^2}{2\sigma_j^2}\right] dw \\
&= -\exp\left[-\rho\left(m_j - \frac{\rho}{2}\sigma^2 - c\left(a_j\right)\right)\right] \\
&\quad \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{\left(w - m_j + \rho\sigma^2\right)^2}{2\sigma_j^2}\right] dw
\end{aligned}
$$

This is simply the utility for a certainty equivalent times one.

$$
U\left[CE\left(a_j\right)\right] \int_{-\infty}^{\infty} f\left(w - \rho\sigma^2 \mid a_j\right) dw = 1
$$

$$
U\left[CE\left(a_j\right)\right] = -\exp\left[-\rho\left(m_j - \frac{\rho}{2}\sigma^2 - c\left(a_j\right)\right)\right]
$$

and

$$\int_{-\infty}^{\infty} f\left(w-\rho\sigma^2 \mid a_j\right) dw = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{\left(w-m_j+\rho\sigma^2\right)^2}{2\sigma_j^2}\right] dw = 1$$

Accordingly, we work with the agent's certainty equivalent to identify the agent's equilibrium input strategy, a_j, for $j = L, H$.

$$\begin{aligned} CE\left(a_j\right) &= E\left[w \mid a_j\right] - Var\left[w \mid a_j\right] - c\left(a_j\right) \\ &= m_j - \frac{\rho}{2}\sigma^2 - c\left(a_j\right) \\ &= w_0 + \delta + \gamma^T \mu_j - \frac{\rho}{2}\gamma^T \Sigma_j \gamma - c\left(a_j\right) \end{aligned}$$

2 Principal's problem

The principal's problem is one of how much to pay for the unobservable quality of the agent's input, a, where we suppose throughout the principal desires input a_H. Suppose there are two performance measures with the following properties.

$$\begin{aligned} y_1 &= a_1 + \alpha a_2 + \varepsilon_1 \\ y_2 &= a_1 + \beta a_2 + \varepsilon_2 \end{aligned}$$

where a_j is the productive benefit of the agent's input for task j, $a_1^H + a_2^H \leq H$ and $a_1^L + a_2^L \leq L$,

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}\right)$$

Having elected a linear (in the performance measures) compensation arrangement with the agent, the principal now chooses the parameters for the fixed wage, δ, and incentive payments, γ.

2.1 without task balance

Accounting for the equilibrium strategy of the agent, the (risk neutral) principal maximizes his/her expected utility.

$$\begin{aligned} &\min_{\delta,\gamma} \quad E\left[I \mid a_H\right] \\ &s.t. \quad CE\left(a_H\right) \geq w_0 + RW \quad (IR) \\ &\qquad\;\; CE\left(a_H\right) \geq CE\left(a_L\right) \quad (IC) \end{aligned}$$

where RW refers to the agent's reservation wage for other employment opportunities. This can be simplified considerably. First, consider the IC constraint where we notice the variance, Σ, is unrelated to the agent's input.

$$\begin{aligned}
CE(a_H) &\geq CE(a_L) \\
w_0 + \delta + \gamma^T \mu_H - \frac{\rho}{2}\gamma^T \Sigma \gamma - c(a_H) &\geq w_0 + \delta + \gamma^T \mu_L - \frac{\rho}{2}\gamma^T \Sigma \gamma - c(a_L) \\
\gamma^T \mu_H - c(a_H) &\geq \gamma^T \mu_L - c(a_L) \\
\gamma^T (\mu_H - \mu_L) &\geq c(a_H) - c(a_L) \\
\gamma^T \begin{bmatrix} a_1^H + \alpha a_2^H - \left(a_1^L + \alpha a_2^L\right) \\ a_1^H + \beta a_2^H - \left(a_1^L + \beta a_2^L\right) \end{bmatrix} &\geq c(a_H) - c(a_L)
\end{aligned}$$

Now, suppose $\alpha, \beta < 1$. Then, naturally the performance measures lead the agent to favor task one and IC simplifies further.

$$\begin{aligned}
\gamma^T \begin{bmatrix} H - L \\ H - L \end{bmatrix} &\geq c(a_H) - c(a_L) \\
\gamma^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} &\geq \frac{c(a_H) - c(a_L)}{H - L} \\
\gamma_1 + \gamma_2 &\geq \frac{c(a_H) - c(a_L)}{H - L} \qquad (IC)
\end{aligned}$$

This constraint is binding at the optimal (linear contract) solution. Hence, we now simply work with $\gamma_1 + \gamma_2 = \frac{c(a_H) - c(a_L)}{H - L}$.

Next, consider IR and solve for the free compensation parameter, δ.

$$\begin{aligned}
CE(a_H) &\geq w_0 + RW \\
w_0 + \delta + \gamma^T \mu_H - \frac{\rho}{2}\gamma^T \Sigma \gamma - c(a_H) &\geq w_0 + RW \\
\delta + \gamma^T \mu_H - \frac{\rho}{2}\gamma^T \Sigma \gamma - c(a_H) &\geq RW \\
\delta &\geq RW + \frac{\rho}{2}\gamma^T \Sigma \gamma + c(a_H) - \gamma^T \mu_H
\end{aligned}$$

Hence, the fixed wage portion of the employment contract, δ, equals the reservation wage plus a risk premium plus the agent's personal cost for input H less expected incentive payments. This constraint is also binding for the optimal (linear contract) solution, $\delta = RW + \frac{\rho}{2}\gamma^T \Sigma \gamma + c(a_H) - \gamma^T \mu_H$.

Now, reframe the principal's problem by substituting the (binding) constraints into the objective function.

$$\begin{aligned}
E[I \mid a_H] &= \delta + \gamma^T \mu_H \\
&= RW + \frac{\rho}{2}\gamma^T \Sigma \gamma + c(a_H) - \gamma^T \mu_H + \gamma^T \mu_H \\
&= RW + \frac{\rho}{2}\gamma^T \Sigma \gamma + c(a_H)
\end{aligned}$$

Then, the principal's optimization problem is

$$\begin{aligned}
\min_{\gamma} \quad & RW + \tfrac{\rho}{2}\gamma^T \Sigma \gamma + c(a_H) \\
s.t. \quad & \gamma_1 + \gamma_2 = \tfrac{c(a_H) - c(a_L)}{H - L}
\end{aligned}$$

but RW, $c(a_H)$, $\frac{\rho}{2}$ are nonnegative constants that have no affect on the choice of γ so they can be ignored and the problem further simplified to minimize the variance of the agent's (incentive) compensation subject to IC.

$$\begin{array}{cc} \min\limits_{\gamma} & \gamma^T \Sigma \gamma \\ s.t. & \gamma_1 + \gamma_2 = \frac{c(a_H) - c(a_L)}{H-L} \end{array}$$

However, this also can be substituted into the objective function to give a one variable optimization problem.

$$\min_{\gamma_1} \begin{bmatrix} \gamma_1 & \frac{c(a_H)-c(a_L)}{H-L} - \gamma_1 \end{bmatrix} \Sigma \begin{bmatrix} \gamma_1 \\ \frac{c(a_H)-c(a_L)}{H-L} - \gamma_1 \end{bmatrix}$$

On solving the first order condition for γ, γ can be substituted to find δ from $\delta = RW + \frac{\rho}{2}\gamma^T \Sigma \gamma + c(a_H) - \gamma^T \mu_H$ and the problem without task balance is resolved.

2.2 task balance

Suppose the agent has no preference with regard to tasks (as above) but the principal desires the agent to balance the two tasks. With performance measures, y, the agent's equilibrium behavior with regard to tasks is

$$\begin{array}{cc} \max\limits_{a_1, a_2} & \gamma_1 (a_1 + \alpha a_2) + \gamma_2 (a_1 + \beta a_2) \\ s.t. & a_1 + a_2 \leq H \end{array}$$

For tasks to be balance, $a_1 = a_2 = \frac{H}{2}$, the principal will need to carefully choose γ such that $(\gamma_1 + \gamma_2) a_1$ balances $(\gamma_1 \alpha + \gamma_2 \beta) a_2$ where $a_1 = a_2$. This, of course, entails

$$\gamma_1 + \gamma_2 = \gamma_1 \alpha + \gamma_2 \beta \qquad (TB)$$

With two performance measures and two constraints controlling γ, IC and TB, if a feasible solution exists degrees of freedom are exhausted and the associated risk premium may be substantially larger than the case without task balance. That is, solving

$$\begin{aligned} \gamma_1 + \gamma_2 &= \frac{c(a_H) - c(a_L)}{H - L} && (IC) \\ \gamma_1 + \gamma_2 &= \gamma_1 \alpha + \gamma_2 \beta && (TB) \end{aligned}$$

leaves no freedom for selecting γ to minimize the agent's risk premium or variance of compensation $\gamma^T \Sigma \gamma$.

3 Performance measure design

The above conclusion reinforces the critical and subtle nature of performance measure design when the principal desires to supply explicit incentives. Naturally balanced performance measures, such as those characterized by $\alpha = \beta = 1$, are extremely challenging to identify. More generally, balanced performance measures are those for which TB imposes no restriction on selection of γ. A general solution is design performance measures with the following properties

$$\beta = \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}{\sigma_1^2 - \sigma_{12}} - \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 - \sigma_{12}}\alpha$$

For example, suppose $\sigma_1^2 = \sigma_2^2$ and $\sigma_{12} = 0$ then $\beta = 2 - \alpha$ produces balanced performance measures, y_1 and y_2. Clearly, designing performance measures, or in other words, supplying explicit incentives presents significant challenges.