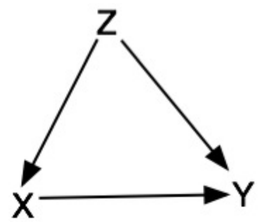
SCM and inverse propensity score weighting

Back-door adjustment

In the DAG below Z is a back-door to the causal effect of X on Y.



Identification of the causal effect requires a back-door adjustment.

$$P(Y=y \mid do(X=x)) = \sum_z P(Y=y \mid x,z)\, P(z)$$

To help cope with many possible levels of Z, the back-door adjustment can be written as an inverse propensity score of the joint distribution of outcome Y, the causal variable X, and the confounders Z where the propensity score is $p \equiv P(X=1 \mid z)$.

$$P(Y=y \mid do(X=x)) = \sum_z \frac{P(Y=y \mid x,z)\, P(z)\, P(x \mid z)}{P(x \mid z)}$$

$$= \sum_z \frac{P(y,x,z)}{P(x \mid z)}$$

Also, we can write the conditional causal effect in similar fashion.

$$P(Y=y \mid do(X=x), Z=z) = P(y \mid x,z)$$

$$= \frac{P(y,x \mid z)}{P(x \mid z)}$$

Then, when the intervention or action is $do(X=1)$ we have

$$P(y \mid do(X=1), z) = \frac{P(y, X=1 \mid z)}{P(X=1 \mid z)} = \frac{P(y, X=1 \mid z)}{p}$$

and when the action is $do(X=0)$ we have

$$P(y \mid do(X=0), z) = \frac{P(y, X=0 \mid z)}{P(X=0 \mid z)} = \frac{P(y, X=0 \mid z)}{1-p}$$

Expected treatment effect

Further, the expected conditional or regression causal effect when the action is $do(X = 1)$ is

$$E[Y \mid do(X = 1), z] = \sum_y \frac{P(y, X = 1 \mid z)}{P(X = 1 \mid z)} y = \sum_y \frac{P(y, X = 1 \mid z)}{p} y$$

Likewise, the expected conditional causal effect when the action is $do(X = 0)$ is

$$E[Y \mid do(X = 0), z] = \sum_y \frac{P(y, X = 0 \mid z)}{P(X = 0 \mid z)} y = \sum_y \frac{P(y, X = 0 \mid z)}{1 - p} y$$

and the conditional expected treatment effect is

$$\begin{aligned} ETE(z) &= E[Y \mid do(X = 1), z] - E[Y \mid do(X = 0), z] \\ &= \sum_y \frac{P(y, X = 1 \mid z)}{P(X = 1 \mid z)} y - \frac{P(y, X = 0 \mid z)}{P(X = 0 \mid z)} y \\ &= \sum_y \frac{P(y, X = 1 \mid z)}{p} y - \frac{P(y, X = 0 \mid z)}{1 - p} y \end{aligned}$$

Then, the unconditional expected treatment effect is

$$\begin{aligned} ETE &= E_z[E[Y \mid do(X = 1), z] - E[Y \mid do(X = 0), z]] \\ &= \sum_z \left(\sum_y \frac{P(y, X = 1 \mid z)}{P(X = 1 \mid z)} y - \frac{P(y, X = 0 \mid z)}{P(X = 0 \mid z)} y \right) P(z) \\ &= \sum_z \sum_y \frac{P(y, X = 1, z)}{P(X = 1 \mid z)} y - \frac{P(y, X = 0, z)}{P(X = 0 \mid z)} y \\ &= \sum_z \sum_y \frac{P(y, X = 1, z)}{p} y - \frac{P(y, X = 0, z)}{1 - p} y \end{aligned}$$

Expected treatment effect on the treated

Similar arguments apply if we wish to focus on only the treated rather than the entire population. The expected treatment effect on the treated is

$$\begin{aligned} ETT &= E\left[Y_{do(X=1)} \mid X = 1\right] - E\left[Y_{do(X=0)} \mid X = 1\right] \\ &\equiv E[Y_1 \mid X = 1] - E[Y_0 \mid X = 1] \end{aligned}$$

The first term is observed, consistency (and do-calculus) insists

$$E[Y_1 \mid X = 1] = E[Y \mid X = 1]$$

$$= \sum_z \sum_y \frac{\Pr(y, X=1, z)}{\Pr(X=1)} y$$

The second term is counterfactual. The law of total probability (also iterated expectations) gives

$$E[Y_0] = E[Y_0 \mid X=1] \Pr(X=1) + E[Y_0 \mid X=0] \Pr(X=0)$$

$$= E[Y_0 \mid X=1] \Pr(X=1) + E[Y \mid X=0] \Pr(X=0)$$

Then,

$$E[Y_0 \mid X=1] = \frac{E[Y_0] - E[Y \mid X=0] \Pr(X=0)}{\Pr(X=1)}$$

By the back-door adjustment

$$E[Y_0] = \sum_z E[Y \mid X=0, z] \Pr(z)$$

$$= \sum_z \sum_y \Pr(y \mid X=0, z) \Pr(z) \, y$$

$$= \sum_z \sum_y \frac{\Pr(y \mid X=0, z) \Pr(X=0 \mid z) \Pr(z)}{\Pr(X=0 \mid z)} y$$

$$= \sum_z \sum_y \Pr(y, X=0, z) \frac{y}{\Pr(X=0 \mid z)}$$

Also,

$$E[Y \mid X=0] \Pr(X=0) = \sum_z \sum_y \Pr(y, X=0, z) \, y$$

Substitution provides

$$E[Y_0 \mid X=1] =$$

$$\left\{ \sum_z \sum_y \Pr(y, X=0, z) \frac{y}{\Pr(X=0 \mid z)} - \Pr(y, X=0, z) \, y \right\} / \Pr(X=1)$$

$$= \frac{\sum_z \sum_y \Pr(y, X=0, z) \frac{\Pr(X=1|z) y}{\Pr(X=0|z)}}{\Pr(X=1)}$$

Putting everything together yields

$$ETT =$$

$$\left\{ \sum_z \sum_y \Pr(y, X=1, z) \, y - \Pr(y, X=0, z) \frac{\Pr(X=1 \mid z) \, y}{\Pr(X=0 \mid z)} \right\} / \Pr(X=1)$$

Treated observations are unscaled by the propensity score while untreated observations are scaled by the propensity score over its complement for the expected treatment on the treated.

By analogy, expected treatment on the untreated is

$$ETU =$$

$$\left\{ \sum_z \sum_y \Pr(y, X=1, z) \frac{\Pr(X=0 \mid z)\, y}{\Pr(X=1 \mid z)} - \Pr(y, X=0, z)\, y \right\} / \Pr(X=0)$$

The expected treatment effect is the probability weighted average of the expected treatment effects on the treated and untreated.

$$ETE = ETT \Pr(X=1) + ETU \Pr(X=0)$$

$$= \sum_z \sum_y \Pr(y, X=1, z)\, y - \Pr(y, X=0, z) \frac{\Pr(X=1 \mid z)}{\Pr(X=0 \mid z)} y$$

$$+ \sum_z \sum_y \Pr(y, X=1, z) \frac{\Pr(X=0 \mid z)\, y}{\Pr(X=1 \mid z)} - \Pr(y, X=0, z)\, y$$

$$= \sum_z \sum_y \Pr(y, X=1, z) \frac{y}{p} - \sum_z \sum_y \Pr(y, X=0, z) \frac{y}{1-p}$$

This matches the earlier result for ETE where treated observations are scaled by the propensity score and untreated observations are scaled by the complement to propensity score.

Example

Finally, we consider a simple, but illustrative Simpson's paradox example. The joint distibution is

$\Pr(y, x, z)$	$X=0, Z=0$	$X=0, Z=1$	$X=1, Z=0$	$X=1, Z=1$
$Y=0$	$\frac{1}{16}$	$\frac{3}{40}$	$\frac{57}{400}$	$\frac{3}{400}$
$Y=1$	$\frac{1}{16}$	$\frac{3}{10}$	$\frac{133}{400}$	$\frac{7}{400}$

Marginal probabilities are $\Pr(Y=1) = \frac{57}{80}, \Pr(X=1) = \frac{1}{2}, \Pr(Z=1) = \frac{2}{5}$. Propensity scores are $\Pr(X=1 \mid Z=0) = \frac{19}{24}$ and $\Pr(X=1 \mid Z=1) = \frac{1}{16}$. Conditional expected values are

$$E[Y \mid X=0] = 0.725$$
$$E[Y \mid X=1] = 0.7$$

Expected treatment effects are

$$ETE = 0.08$$
$$ETT = 0.185$$
$$ETU = -0.025$$