# Information synergy, part 3:
belief updating

These notes describe belief updating for dynamic Kelly-Ross investments where initial conditions may matter. This note differs from the first two notes on information synergy in that decision maker's are uncertain regarding state probability assignment, that is, we have a probability distribution over a probability distribution (as with the beta or its multivariate extension, the Dirichlet distribution). Hence, information is employed to update beliefs regarding state probabilities and accordingly investment strategies. First, we prove a generalization of the Kelly strategy to the setting in which there is uncertainty regarding state probability assignment. Then, we explore uninformed belief updating before incorporating probability assignment based on information in state prices, that is, probability distributions reflect Ross' recovery theorem and beliefs are updated. Updating information includes coarse and/or late accounting information, $z_a$, and other finer information, $z$, that may be biased or unbiased.

Asset returns may be independent of the initial state, we refer to this as the static information case. On the other hand, asset returns may depend on the initial state (initial conditions). We refer to this as the dynamic information case.

## 1 Uncertainty over state probability assignment

Probability assignment is a state of knowledge exercise (Jaynes 2003). If the individual is uncertain about state probabilities then assigning a distribution to describe such uncertainty is natural. In these notes, we work with Dirichlet probability assignment but the proof below is satisfied for any distribution over probabilities (bounded between zero and one) for which the mean exists.[1]

**Lemma 1.1 (Kelly criterion with uncertain state probabilities)** *Suppose the conditions of the Kelly criterion are satisfied (sufficient alternatives to span the states and positive state prices, y) and we wish to maximize expected long-run wealth (the Kelly criterion) but state probabilities are uncertain. Then, the optimal fraction of wealth invested in the ith Arrow-Debreu portfolio, $k_i$, equals the expected (or mean) state probability, $E[p_i]$.*

**Proof.** Expected long-run wealth maximization when state probabilities are uncertain is

$$\begin{array}{ll} \max\limits_{k} & E_p\left[\sum_{i=1}^n p_i \log \frac{k_i}{y_i}\right] \\ s.t. & \sum_{i=1}^n k_i \leq 1 \end{array}$$

The Lagrangian is

$$E_p\left[\sum_{i=1}^n p_i \log \frac{k_i}{y_i}\right] - \lambda\left(\sum_{i=1}^n k_i - 1\right)$$

[1] The Dirichlet distribution is the natural or maximum entropy probability distribution based on the moment conditions $E[\log p_i]$ for all $i$.

First order conditions are

$$\frac{E[p_i]}{k_i} - \lambda = 0 \quad \forall i$$

Since both $\sum_{i=1}^{n} k_i = 1$ and $\sum_{i=1}^{n} E[p_i] = 1$, $\lambda = 1$ and $k_i = E[p_i]$, $\forall i$. ■

# 2 Static information

First, we consider a simple static information setting.

**Example 2.1 (unknown static distribution)** *Suppose returns are identified by $A$ for three projects (in rows) and three states (in columns) and normalized investment $x$*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1.1 & \frac{1}{1.1} & 1 \\ \frac{1}{1.1} & 1 & 1.1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

*then state prices are* $y = \begin{bmatrix} 0.332326 \\ 0.365559 \\ 0.302115 \end{bmatrix}$.

*Suppose the only background knowledge we utilize is that there are three states,*[2] *then probabilities are assigned a Dirichlet distribution with parameters equal to one.*[3]

$$p \sim Dir\left(a_1 = 1, a_2 = 1, a_3 = 1\right)$$

*with density function*

$$f(p) = \frac{\Gamma(a_1 + a_2 + a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} p_1^{a_1 - 1} p_2^{a_2 - 1} (1 - p_1 - p_2)^{a_3 - 1}, 0 < p_i < 1$$

$$f(p; a_1 = a_2 = a_3 = 1) = 2$$

*uniformly distributed and* $E[p] = \begin{bmatrix} \frac{a_1}{a_1 + a_2 + a_3} \\ \frac{a_2}{a_1 + a_2 + a_3} \\ \frac{a_3}{a_1 + a_2 + a_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

The next example involves updating that acknowledges information contained in equilibrium prices. The information structure can be derived from state prices following Ross' recovery theorem.

[2] We return to this example in the next section and utilize Ross' recovery theorem to assign probabilities.

[3] $a_j = 1$ is the maximum entropy Dirichlet distribution while $a_j = \frac{1}{2}$ corresponds to Jeffrey's uninformed prior conjugate to a multinomial likelihood function. Entropy is greater for $a_j = 1$ than for $a_j = \frac{1}{2}$ for all $j$. Entropy for the Dirichlet distribution is $H(a) = \log \frac{\Gamma(a_1)\cdots\Gamma(a_n)}{\Gamma(a_0)} + (a_0 - n) \frac{\partial \log \Gamma(a_0)}{\partial a_0} - \sum_{j=1}^{n} (a_j - 1) \frac{\partial \log \Gamma(a_j)}{\partial a_j}$ where $a_0 = \sum_{j=1}^{n} a_j$. $H(a)$ is maximized for $a_j = 1$ for all $j$.

**Example 2.2 (static equilibrium price-based distribution)** *Suppose everything remains as in example 2.1 except state probabilities are updated based on Ross' recovery theorem. Recall, there are three assets with known returns in three states*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1.1 & \frac{1}{1.1} & 1 \\ \frac{1}{1.1} & 1 & 1.1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

*and* $y = \begin{bmatrix} 0.332326 \\ 0.365559 \\ 0.302115 \end{bmatrix}$.

*As returns are static (don't depend on the initial state), the state price matrix involves the same row repeated*

$$P = \begin{bmatrix} 0.332326 & 0.365559 & 0.302115 \\ 0.332326 & 0.365559 & 0.302115 \\ 0.332326 & 0.365559 & 0.302115 \end{bmatrix}$$

*and since the riskless rate is zero, state transition probabilities are equal to* $P$.

$$\begin{aligned} F &= \frac{1}{\delta} D P D^{-1} \\ &= \frac{1}{1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.332326 & 0.365559 & 0.302115 \\ 0.332326 & 0.365559 & 0.302115 \\ 0.332326 & 0.365559 & 0.302115 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0.332326 & 0.365559 & 0.302115 \\ 0.332326 & 0.365559 & 0.302115 \\ 0.332326 & 0.365559 & 0.302115 \end{bmatrix} \end{aligned}$$

*Uninformed (uniform) priors can be updated in accordance with this equilibrium price evidence via a likelihood function*

$$\ell \left( p_i \mid P \right) \propto p_1^{\alpha F_{i1} - 1} \cdots p_n^{\alpha F_{in} - 1}$$

*to generate a Dirichlet posterior distribution for initial state i probabilities* $p_i$

$$\Pr \left( p_i \mid P \right) \propto p_1^{\alpha F_{i1} - 1} \cdots p_n^{\alpha F_{in} - 1}$$

*with expected value*

$$\Pr \left( p_i \mid P \right) = \begin{bmatrix} \frac{F_{i1}}{F_{i1} + \cdots + F_{in}} \\ \vdots \\ \frac{F_{in}}{F_{i1} + \cdots + F_{in}} \end{bmatrix}$$

*where* $\alpha$ *is a concentration parameter indicating the strength of beliefs in the evidence (large* $\alpha$ *indicates strong beliefs while small* $\alpha$ *indicates weak beliefs)*[4].

[4] In subsequent sections, we frequently refer to this Ross' recovery theorem distribution as the prior when updating based on accounting and/or other information.

*Since* $\delta = 1$, $E[r] = \log\frac{1}{1} = 0$ *and the expected long-run growth rate is* $\exp(0) = \frac{1}{\delta} = 1$ *an increase of* $0.00302115$ *over the uninformed case.*[5] *This richer probability assignment appears to generate lower expected returns than the naive (or uninformed) assignment in example 2.1. However, the expected return associated with a uniform investment on the three Arrow-Debreu portfolios is*

$$\begin{aligned} E[r \mid ignored] &= \begin{bmatrix} 0.332326 & 0.365559 & 0.302115 \end{bmatrix} \\ &\quad \times \log\left(\begin{bmatrix} \frac{1}{0.332326} & 0 & 0 \\ 0 & \frac{1}{0.365559} & 0 \\ 0 & 0 & \frac{1}{0.302115} \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}\right) \\ &= -0.00302115 \end{aligned}$$

*or an expected growth rate equal to* $\exp(-0.00302115) = 0.996983$*. The difference arises from probability beliefs and their implication for expectations. If we ignore the information in state prices for assigning probabilities and update as in example  where draws arise consistent with steady-state probabilities*

$$p_{ss}^T = \begin{bmatrix} 0.332326 & 0.365559 & 0.302115 \end{bmatrix}$$

*then eventually the updating will reflect the recovery theorem probability assignment (provided no shift in equilibrium occurs).*

*Standard updating involves realization* $s_i$ *and updating via a multinomial likelihood function which combines with the Dirichlet(*$a_1 = 1, a_2 = 1, a_3 = 1$*) prior distribution to form a Dirichlet posterior distribution with parameters* $(a_1 + s_1, a_2 + s_2, a_3 + s_3)$ *and density function*

$$f(p) = \frac{1}{\frac{\Gamma(a_1+s_1)\Gamma(a_2+s_2)\Gamma(a_3+s_3)}{\Gamma(a_1+s_1+a_2+s_2+a_3+s_3)}} p_1^{a_1+s_1-1} p_2^{a_2+s_2-1} (1 - p_1 - p_2)^{a_3+s_3-1}$$

*where* $s_j$ *refers to the number of realized occurrences in state* $j$*. This posterior distribution has mean* $E[p \mid s] = \begin{bmatrix} \frac{a_1+s_1}{a_1+s_1+a_2+s_2+a_3+s_3} \\ \frac{a_2+s_2}{a_1+s_1+a_2+s_2+a_3+s_3} \\ \frac{a_3+s_3}{a_1+s_1+a_2+s_2+a_3+s_3} \end{bmatrix}$ *and rebalanced weights on the Arrow-Debreu portfolios* $k_t = E[p \mid s]$*.*

*For instance,* $n = 10$ *draws produce frequencies* $\begin{bmatrix} 3 & 4 & 3 \end{bmatrix}$ *with posterior probability*

$$\frac{1}{\frac{\Gamma(4)\Gamma(5)\Gamma(4)}{\Gamma(4+5+4)}} p_1^3 p_2^4 (1 - p_1 - p_2)^3$$

[5] Weights on the nominal projects are

$$\begin{aligned} w^T &= k^T \Omega A^{-1} \\ &= E[p]^T \Omega A^{-1} \\ &= \begin{bmatrix} -1.00303 & 0.969697 & 1.03333 \end{bmatrix} \end{aligned}$$

where $w^T \iota = 1$ for $\iota$ a vector of ones. Further, nominal portfolio returns are unique state-by-state $w^T A = \begin{bmatrix} 1.00303 & 0.911846 & 1.10333 \end{bmatrix}$ and $E[r] = E[p]^T \log(A^T w) = 0.00302572$.

*and expected state probabilities*

$$\left[\begin{array}{ccc} \frac{4}{13} \approx 0.307692 & \frac{5}{13} \approx 0.384615 & \frac{4}{13} \approx 0.307692 \end{array}\right]$$

$$\begin{aligned} E\left[r \mid n=10\right] &= \left[\begin{array}{ccc} 0.332326 & 0.365559 & 0.302115 \end{array}\right] \\ &\quad \times \log\left(\left[\begin{array}{ccc} \frac{1}{0.332326} & 0 & 0 \\ 0 & \frac{1}{0.365559} & 0 \\ 0 & 0 & \frac{1}{0.302115} \end{array}\right]\left[\begin{array}{c} \frac{4}{13} \\ \frac{5}{13} \\ \frac{4}{13} \end{array}\right]\right) \\ &= -0.00149174 \end{aligned}$$

*while* $n = 100$ *draws produce frequencies* $\left[\begin{array}{ccc} 33 & 37 & 30 \end{array}\right]$ *with posterior probability*

$$\frac{1}{\frac{\Gamma(34)\Gamma(38)\Gamma(31)}{\Gamma(34+38+31)}} p_1^{33} p_2^{38} \left(1 - p_1 - p_2\right)^{30}$$

*and expected state probabilities*

$$\left[\begin{array}{ccc} \frac{34}{103} \approx 0.330097 & \frac{38}{103} \approx 0.368932 & \frac{31}{103} \approx 0.300971 \end{array}\right]$$

$$\begin{aligned} E\left[r \mid n=100\right] &= \left[\begin{array}{ccc} 0.332326 & 0.365559 & 0.302115 \end{array}\right] \\ &\quad \times \log\left(\left[\begin{array}{ccc} \frac{1}{0.332326} & 0 & 0 \\ 0 & \frac{1}{0.365559} & 0 \\ 0 & 0 & \frac{1}{0.302115} \end{array}\right]\left[\begin{array}{c} \frac{34}{103} \\ \frac{38}{103} \\ \frac{31}{103} \end{array}\right]\right) \\ &= -0.0000251487 \end{aligned}$$

*and so on. Of course, the expected loss associated with ignoring the information imbedded in state prices diminishes with each updating.*

Suppose the agent employs priors based on equilibrium prices but accommodates evidence indicating an equilibrium shift or equivalently a shift in probability assignment. Such evidence could be a realized shift in relative prices or, if the agent is out front with private information, a foretelling of a shift in relative prices, or simply private information regarding the post-transition state with relative prices unaltered. Depending on the agent's perception (this is a state of knowledge matter) of the relative strength of the likelihood $\sum_j s_j$ to the prior $\sum_j a_j$, the evidence may dominate the posterior, be dominated by the prior, or a blend of the two. We consider $\sum_j a_j = n$ weak priors, $\sum_j a_j = 10n$ moderate priors, and $\sum_j a_j = 100n$ strong priors where proportions indicated by equilibrium prices and Ross' recovery theorem are maintained.

**Example 2.3 (updating from an equilibrium price-based prior)** *Suppose everything remains as in example 2.1 except the agent employs priors based on Ross' recovery theorem. Recall, there are three assets with known returns in three states*

$$A = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1.1 & \frac{1}{1.1} & 1 \\ \frac{1}{1.1} & 1 & 1.1 \end{array}\right], x = \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

*and state prices* $y = \begin{bmatrix} 0.332326 \\ 0.365559 \\ 0.302115 \end{bmatrix}$. *Therefore, the agent's prior employs* $(a_1 : a_2 : a_3)$ *as* $(0.332326 : 0.365559 : 0.302115)$*. As new evidence arrives the agent updates probability beliefs regarding future state transitions.*

*Weak priors (say,* $\alpha = 3$*) are*

$$\propto p_1^{-0.003022} p_2^{0.096677} (1 - p_1 - p_2)^{-0.093655}$$

*while moderate priors (say,* $\alpha = 30$*) are*

$$\propto p_1^{8.96978} p_2^{9.96677} (1 - p_1 - p_2)^{8.06345}$$

*and strong priors (say,* $\alpha = 300$*) are*

$$\propto p_1^{98.6978} p_2^{108.668} (1 - p_1 - p_2)^{89.6345}$$

*Suppose no shift in equilibrium occurs but evidence dribbles in initially suggesting a change. Of course, any shift in the agent's beliefs depend on the relative strength of assigned priors. Following* $z_1 = s_1$*, a weak prior leads to the posterior belief*

$$\propto p_1^{1-0.003022} p_2^{0.096677} (1 - p_1 - p_2)^{-0.093655}$$

*with expected probabilities*

$$E[p \mid z_1 = s_1] = \begin{bmatrix} 0.499245 & 0.274169 & 0.226586 \end{bmatrix}$$

*while a moderate prior leads to the posterior*

$$\propto p_1^{1+8.96978} p_2^{9.96677} (1 - p_1 - p_2)^{8.06345}$$

*with expected probabilities*

$$E[p \mid z_1 = s_1] = \begin{bmatrix} 0.353864 & 0.353767 & 0.292369 \end{bmatrix}$$

*and a strong prior leads to the posterior*

$$\propto p_1^{1+98.6978} p_2^{108.668} (1 - p_1 - p_2)^{89.6345}$$

*with expected probabilities*

$$E[p \mid z_1 = s_1] = \begin{bmatrix} 0.334544 & 0.364345 & 0.301111 \end{bmatrix}$$

*As expected, a weak prior is decidedly sensitive to the evidence while moderate and strong priors are altered relatively little by the same evidence. Below we* *tabulate the impact of various evidence on weak, moderate, and strong priors where the evidence eventually converges toward the steady-state likelihoods.*

| evidence, $z$ | | | $E[p \mid z]$ | | |
|---|---|---|---|---|---|
| $s_1$ | $s_2$ | $s_3$ | *weak priors* | *moderate priors* | *strong priors* |
| 1 | 0 | 0 | $\begin{bmatrix} 0.499245 \\ 0.274169 \\ 0.226586 \end{bmatrix}$ | $\begin{bmatrix} 0.353864 \\ 0.353767 \\ 0.292369 \end{bmatrix}$ | $\begin{bmatrix} 0.334544 \\ 0.364345 \\ 0.301111 \end{bmatrix}$ |
| 2 | 0 | 0 | $\begin{bmatrix} 0.599396 \\ 0.219335 \\ 0.181269 \end{bmatrix}$ | $\begin{bmatrix} 0.374056 \\ 0.342712 \\ 0.283233 \end{bmatrix}$ | $\begin{bmatrix} 0.336748 \\ 0.363138 \\ 0.300114 \end{bmatrix}$ |
| 3 | 0 | 0 | $\begin{bmatrix} 0.666163 \\ 0.18278 \\ 0.151058 \end{bmatrix}$ | $\begin{bmatrix} 0.393024 \\ 0.332326 \\ 0.27465 \end{bmatrix}$ | $\begin{bmatrix} 0.338937 \\ 0.36194 \\ 0.299124 \end{bmatrix}$ |
| 3 | 4 | 3 | $\begin{bmatrix} 0.30746 \\ 0.392052 \\ 0.300488 \end{bmatrix}$ | $\begin{bmatrix} 0.324245 \\ 0.374169 \\ 0.301586 \end{bmatrix}$ | $\begin{bmatrix} 0.331283 \\ 0.36667 \\ 0.302047 \end{bmatrix}$ |
| 33 | 37 | 30 | $\begin{bmatrix} 0.330068 \\ 0.369871 \\ 0.300062 \end{bmatrix}$ | $\begin{bmatrix} 0.330537 \\ 0.368975 \\ 0.300488 \end{bmatrix}$ | $\begin{bmatrix} 0.331745 \\ 0.366669 \\ 0.301586 \end{bmatrix}$ |

*Now, suppose a substantial shift in equilibrium occurs. The amount of evidence required to shift the agent's beliefs depends on the relative strength of the agent's priors. Suppose state-transition probabilities shift to* $p^r = \begin{bmatrix} 0.8 & 0.1 & 0.1 \end{bmatrix}$*. If the agent shifts state-probability assignment accordingly. The implied state prices (maintaining preferences as before) are also* $y^r = \begin{bmatrix} 0.8 & 0.1 & 0.1 \end{bmatrix}$*.*

*Further, in this static setting we can deduce the updated return profile (essentially revised equilibrium prices) by revising each row of* $A$ *as* $\frac{A_j}{A_j y^r}$ *where* $A_j y^r$ *is effectively the revised equilibrium (relative) price.*

$$A^r = \begin{bmatrix} 1 & 1 & 1 \\ \frac{1.1}{1.070909} & \frac{1}{1.1(1.070909)} & \frac{1}{1.070909} \\ \frac{1}{1.1(0.9372727)} & \frac{1}{0.9372727} & \frac{1.1}{0.9372727} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1.0271647 & 0.8488964 & 0.9337861 \\ 0.9699321 & 1.0669253 & 1.1736178 \end{bmatrix}$$

*Below we tabulate the impact of various evidence on weak, moderate, and* *strong priors where the evidence converges toward the steady-state likelihoods.*

| *evidence,* $z$ | | | $E[p \mid z]$ | | |
|---|---|---|---|---|---|
| $s_1$ | $s_2$ | $s_3$ | *weak priors* | *moderate priors* | *strong priors* |
| 8 | 1 | 1 | $\begin{bmatrix}0.692075\\0.161283\\0.146642\end{bmatrix}$ | $\begin{bmatrix}0.449245\\0.299169\\0.251586\end{bmatrix}$ | $\begin{bmatrix}0.347412\\0.356993\\0.295595\end{bmatrix}$ |
| 16 | 2 | 2 | $\begin{bmatrix}0.738999\\0.134638\\0.126363\end{bmatrix}$ | $\begin{bmatrix}0.519396\\0.259335\\0.221269\end{bmatrix}$ | $\begin{bmatrix}0.361556\\0.348962\\0.289483\end{bmatrix}$ |
| 32 | 4 | 4 | $\begin{bmatrix}0.767372\\0.118527\\0.114101\end{bmatrix}$ | $\begin{bmatrix}0.599568\\0.213811\\0.186621\end{bmatrix}$ | $\begin{bmatrix}0.387346\\0.334317\\0.278337\end{bmatrix}$ |
| 64 | 8 | 8 | $\begin{bmatrix}0.783096\\0.109599\\0.107305\end{bmatrix}$ | $\begin{bmatrix}0.672453\\0.172425\\0.155122\end{bmatrix}$ | $\begin{bmatrix}0.430784\\0.309652\\0.259564\end{bmatrix}$ |
| 80 | 10 | 10 | $\begin{bmatrix}0.786378\\0.107735\\0.105887\end{bmatrix}$ | $\begin{bmatrix}0.692075\\0.161283\\0.146642\end{bmatrix}$ | $\begin{bmatrix}0.449245\\0.299169\\0.251586\end{bmatrix}$ |
| 960 | 120 | 120 | $\begin{bmatrix}0.798834\\0.100662\\0.100504\end{bmatrix}$ | $\begin{bmatrix}0.788593\\0.106477\\0.10493\end{bmatrix}$ | $\begin{bmatrix}0.706465\\0.153112\\0.140423\end{bmatrix}$ |

# 3 Dynamic information and state belief updating

A more dynamic setting involves payoffs/returns possibly including riskless returns that depend on the initial state. For instance, it may be that riskless returns are the same across numerous initial states but when conditions change sufficiently so does the riskless return. Hence, in the remaining examples we'll explore this richer return profile in which initial conditions (the initial state) may matter along with biased nonaccounting information and/or coarse accounting information.

Next, we explore accounting and other information regarding transition states that refine transition probabilities assigned via Ross' recovery theorem given the initial state. As the initial state is known, we focus on updating from a single information signal. In particular, we compare noisy other information with imperfect coarse accounting information for the foregoing four state-four project setting.

## 3.1 Likelihood functions

First, we employ likelihood functions for accounting and other information that largely dominate priors but nonetheless produce Dirichlet posterior distributions. While nonstandard, an advantage of these likelihood functions is that strength of priors is muted and the experimental manipulation focuses on the extent of noise in the accounting and other information.[6] The likelihood function for nonaccounting information is

$$\begin{aligned}\ell(s;z) \quad \propto \quad & (1)^{\Im(\emptyset)} \left( p_1^{\frac{a_2+a_3+a_4}{4}} p_2^{-\frac{a_2}{4}} p_3^{-\frac{a_3}{4}} (1-p_1-p_2-p_3)^{-\frac{a_4}{4}} \right)^{\Im(z_1)} \\ & \times \cdots \times \\ & \left( p_1^{-\frac{a_1}{4}} p_2^{-\frac{a_2}{4}} p_3^{-\frac{a_3}{4}} (1-p_1-p_2-p_3)^{\frac{a_1+a_2+a_3}{4}} \right)^{\Im(z_4)}\end{aligned}$$

where $\Im(z_j)$ is an indicator function equal to one when signal $z_j$ is observed and zero otherwise. No information is revealed with probability $\pi = 0.3$. The probability of each state transition signal is then $(1-\pi) F_{ij}$ where $i$ refers to the initial state and $z_j$ is the signal.

Our Dirichlet prior reflects Ross' recovery theorem

$$\Pr(p) \propto p_1^{a_1-1} p_2^{a_2-1} p_3^{a_3-1} (1-p_1-p_2-p_3)^{a_4-1}$$

where the concentration parameters $a_j$ are simply the transition probabilities from the recovery theorem for the given initial state, $F_{ij}$.

Hence, the posterior distribution is

$$\begin{aligned}\Pr(p \mid z) \quad \propto \quad & \left( p_1^{a_1-1} p_2^{a_2-1} p_3^{a_3-1} (1-p_1-p_2-p_3)^{a_4-1} \right)^{\Im(\emptyset)} \\ & \left( p_1^{a_1-1+\frac{a_2+a_3+a_4}{4}} p_2^{\frac{3a_2}{4}-1} p_3^{\frac{3a_3}{4}-1} (1-p_1-p_2-p_3)^{\frac{3a_4}{4}-1} \right)^{\Im(z_1)} \\ & \times \cdots \times \\ & \left( p_1^{\frac{3a_1}{4}-1} p_2^{\frac{3a_2}{4}-1} p_3^{\frac{3a_3}{4}-1} (1-p_1-p_2-p_3)^{a_4-1+\frac{a_1+a_2+a_3}{4}} \right)^{\Im(z_4)}\end{aligned}$$

which follows a Dirichlet distribution with means

$$\begin{aligned}E[p \mid \emptyset] &= \frac{1}{a_1+a_2+a_3+a_4} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \\ E[p \mid z_1] &= \frac{1}{a_1+a_2+a_3+a_4} \begin{bmatrix} a_1 + \frac{a_2+a_3+a_4}{4} \\ \frac{3a_2}{4} \\ \frac{3a_3}{4} \\ \frac{3a_4}{4} \end{bmatrix} \\ & \vdots \\ E[p \mid z_4] &= \frac{1}{a_1+a_2+a_3+a_4} \begin{bmatrix} \frac{3a_1}{4} \\ \frac{3a_2}{4} \\ \frac{3a_3}{4} \\ a_4 + \frac{a_1+a_2+a_3}{4} \end{bmatrix}\end{aligned}$$

[6] In a later section, we employ an alternative set of likelihood functions.

The illustration above employs a noise parameter $b = \frac{1}{4}$ which we hold constant for other information but will vary for accounting information.

Coarse accounting information is noisy in addition to coarse. The coarse accounting information has likelihood function

$$\begin{aligned}\ell(s; z^a) \quad &\propto \quad (1)^{\Im(\emptyset)} \left( p_1^{\frac{a_1 b(a_3+a_4)}{a_1+a_2}} p_2^{\frac{a_2 b(a_3+a_4)}{a_1+a_2}} p_3^{-ba_3} (1-p_1-p_2-p_3)^{-ba_4} \right)^{\Im(z_{12}^a)} \\ &\times \left( p_1^{-ba_1} p_2^{-ba_2} p_3^{\frac{a_3 b(a_1+a_2)}{a_3+a_4}} (1-p_1-p_2-p_3)^{\frac{a_4 b(a_1+a_2)}{a_3+a_4}} \right)^{\Im(z_{34}^a)}\end{aligned}$$

Again, no information is revealed with probability $\pi = 0.3$ and the accounting signals arise with probability $(1-\pi)\left(F_{ij} + F_{i(j+1)}\right)$ where $i$ refers to the initial state and $j = 1$ for $z_{12}^a$ or $j = 3$ for $z_{34}^a$. Combining the likelihood function with above prior yields a Dirichlet posterior distribution

$$\Pr(p \mid z^a) \propto$$

$$\begin{aligned}&\left( p_1^{a_1-1} p_2^{a_2-1} p_3^{a_3-1} (1-p_1-p_2-p_3)^{a_4-1} \right)^{\Im(\emptyset)} \\ &\left( p_1^{a_1-1+\frac{a_1 b(a_3+a_4)}{a_1+a_2}} p_2^{a_2-1+\frac{a_2 b(a_3+a_4)}{a_1+a_2}} p_3^{(1-b)a_3-1} (1-p_1-p_2-p_3)^{(1-b)a_4-1} \right)^{\Im(z_{12}^a)} \\ &\left( p_1^{(1-b)a_1-1} p_2^{(1-b)a_2-1} p_3^{a_3-1+\frac{a_3 b(a_1+a_2)}{a_3+a_4}} (1-p_1-p_2-p_3)^{a_4-1+\frac{a_4 b(a_1+a_2)}{a_3+a_4}} \right)^{\Im(z_{34}^a)}\end{aligned}$$

with means

$$\begin{aligned}E[p \mid \emptyset] &= \frac{1}{a_1+a_2+a_3+a_4} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \\ E[p \mid z_{12}^a] &= \frac{1}{a_1+a_2+a_3+a_4} \begin{bmatrix} a_1 + \frac{a_1 b(a_3+a_4)}{a_1+a_2} \\ a_2 + \frac{a_2 b(a_3+a_4)}{a_1+a_2} \\ (1-b)a_3 \\ (1-b)a_4 \end{bmatrix} \\ &\vdots \\ E[p \mid z_{34}^a] &= \frac{1}{a_1+a_2+a_3+a_4} \begin{bmatrix} (1-b)a_1 \\ (1-b)a_2 \\ a_3 + \frac{a_3 b(a_1+a_2)}{a_3+a_4} \\ a_4 + \frac{a_4 b(a_1+a_2)}{a_3+a_4} \end{bmatrix}\end{aligned}$$

### 3.1.1 Expected gain from information

For noise parameter $b \leq \frac{1}{4}$ clearly other information is preferred. We experiment with $b = \frac{1}{3}$ and $b = \frac{1}{2}$ and compare expected gains from other information with coarse accounting information. Expected gains from information when employing a Kelly investment strategy are again reflected in mutual information, $I(s; z)$.

| | $z\left(b = \frac{1}{4}\right)$ | $z^a\left(b = \frac{1}{3}\right)$ | $z^a\left(b = \frac{1}{2}\right)$ |
|---|---|---|---|
| $I(s; z)$ | 0.0484224 | 0.0388264 | 0.0890673 |

Noisy other information (with noise parameter $b = \frac{1}{4}$) offers greater benefits than coarse accounting information with noise parameter $b = \frac{1}{3}$ but coarse accounting information with noise parameter $b = \frac{1}{2}$ offers significantly greater expected benefits than other information.

Details include posterior probabilities associated with five signals over the four transition-states for each of four initial states or 80 joint probabilities and probability means for other (nonaccounting) information. While accounting information involves three signals over the four transition-states for each of four initial states or 48 joint probabilities and probability means. From the lemma, probability means are the key to implementing a Kelly investment strategy. Posterior mean probabilities are tabulated below by initial state for accounting and other information.

Initial state 1 involves

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.40373 | 0.0486327 | 0.502097 | 0.0455403 |
| $E[\Pr(s \mid z_1)]$ | 0.282611 | 0.552797 | 0.0364746 | 0.376573 | 0.0341552 |
| $E[\Pr(s \mid z_2)]$ | 0.0340429 | 0.302797 | 0.286475 | 0.376573 | 0.0341552 |
| $E[\Pr(s \mid z_3)]$ | 0.351468 | 0.302797 | 0.0364746 | 0.626573 | 0.0341552 |
| $E[\Pr(s \mid z_4)]$ | 0.0318782 | 0.302797 | 0.0364746 | 0.376573 | 0.284155 |

Probabilities for initial state 1 given other information $z\left(b = \frac{1}{4}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.40373 | 0.0486327 | 0.502097 | 0.0455403 |
| $E[\Pr(s \mid z^a_{12})]$ | 0.316654 | 0.56665 | 0.0682579 | 0.334731 | 0.0303602 |
| $E[\Pr(s \mid z^a_{34})]$ | 0.383346 | 0.269153 | 0.0324218 | 0.640346 | 0.0580794 |

Probabilities for initial state 1 given accounting information $z^a\left(b = \frac{1}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.40373 | 0.0486327 | 0.502097 | 0.0455403 |
| $E[\Pr(s \mid z^a_{12})]$ | 0.316654 | 0.648111 | 0.0780705 | 0.251049 | 0.0227701 |
| $E[\Pr(s \mid z^a_{34})]$ | 0.383346 | 0.201865 | 0.0243164 | 0.70947 | 0.064349 |

Probabilities for initial state 1 given accounting information $z^a\left(b = \frac{1}{2}\right)$

Initial state 2 involves

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.326953 | 0.30984 | 0.310455 | 0.0527523 |
| $E[\Pr(s \mid z_1)]$ | 0.228867 | 0.495215 | 0.23238 | 0.232841 | 0.0395642 |
| $E[\Pr(s \mid z_2)]$ | 0.216888 | 0.245215 | 0.48238 | 0.232841 | 0.0395642 |
| $E[\Pr(s \mid z_3)]$ | 0.217319 | 0.245215 | 0.23238 | 0.482841 | 0.0395642 |
| $E[\Pr(s \mid z_4)]$ | 0.0369266 | 0.245215 | 0.23238 | 0.232841 | 0.289564 |

Probabilities for initial state 2 given other information $z \left(b = \frac{1}{4}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.326953 | 0.30984 | 0.310455 | 0.0527523 |
| $E[\Pr(s \mid z_{12}^a)]$ | 0.445755 | 0.389114 | 0.368748 | 0.20697 | 0.0351682 |
| $E[\Pr(s \mid z_{34}^a)]$ | 0.254245 | 0.217969 | 0.20656 | 0.49189 | 0.0835816 |

Probabilities for initial state 2 given accounting information $z^a \left(b = \frac{1}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.326953 | 0.30984 | 0.310455 | 0.0527523 |
| $E[\Pr(s \mid z_{12}^a)]$ | 0.445755 | 0.420195 | 0.398201 | 0.155228 | 0.0263761 |
| $E[\Pr(s \mid z_{34}^a)]$ | 0.254245 | 0.163476 | 0.15492 | 0.582607 | 0.0989962 |

Probabilities for initial state 2 given accounting information $z^a \left(b = \frac{1}{2}\right)$

Initial state 3 involves

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.330195 | 0.323051 | 0.319998 | 0.0267554 |
| $E[\Pr(s \mid z_1)]$ | 0.231137 | 0.497646 | 0.242288 | 0.239999 | 0.0200666 |
| $E[\Pr(s \mid z_2)]$ | 0.226136 | 0.247646 | 0.492288 | 0.239999 | 0.0200666 |
| $E[\Pr(s \mid z_3)]$ | 0.223999 | 0.247646 | 0.242288 | 0.489999 | 0.0200666 |
| $E[\Pr(s \mid z_4)]$ | 0.0187288 | 0.247646 | 0.242288 | 0.239999 | 0.270067 |

Probabilities for initial state 3 given other information $z \left(b = \frac{1}{4}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.330195 | 0.323051 | 0.319998 | 0.0267554 |
| $E[\Pr(s \mid z_{12}^a)]$ | 0.457272 | 0.38862 | 0.380211 | 0.213332 | 0.0178369 |
| $E[\Pr(s \mid z_{34}^a)]$ | 0.242728 | 0.22013 | 0.215367 | 0.520946 | 0.0435568 |

Probabilities for initial state 3 given accounting information $z^a \left(b = \frac{1}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.330195 | 0.323051 | 0.319998 | 0.0267554 |
| $E[\Pr(s \mid z^a_{12})]$ | 0.457272 | 0.417832 | 0.408791 | 0.159999 | 0.0133777 |
| $E[\Pr(s \mid z^a_{34})]$ | 0.242728 | 0.165098 | 0.161525 | 0.621419 | 0.0519575 |

Probabilities for initial state 3 given accounting information $z^a \left(b = \frac{1}{2}\right)$

Initial state 4 involves

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.281848 | 0.356128 | 0.0699203 | 0.292104 |
| $E[\Pr(s \mid z_1)]$ | 0.197294 | 0.461386 | 0.267096 | 0.0524403 | 0.219078 |
| $E[\Pr(s \mid z_2)]$ | 0.249289 | 0.211386 | 0.517096 | 0.0524403 | 0.219078 |
| $E[\Pr(s \mid z_3)]$ | 0.0489442 | 0.211386 | 0.267096 | 0.30244 | 0.219078 |
| $E[\Pr(s \mid z_4)]$ | 0.204473 | 0.211386 | 0.267096 | 0.0524403 | 0.469078 |

Probabilities for initial state 4 given other information $z \left(b = \frac{1}{4}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.281848 | 0.356128 | 0.0699203 | 0.292104 |
| $E[\Pr(s \mid z^a_{12})]$ | 0.446583 | 0.33516 | 0.42349 | 0.0466136 | 0.194736 |
| $E[\Pr(s \mid z^a_{34})]$ | 0.253417 | 0.187899 | 0.237418 | 0.110993 | 0.46369 |

Probabilities for initial state 4 given accounting information $z^a \left(b = \frac{1}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.281848 | 0.356128 | 0.0699203 | 0.292104 |
| $E[\Pr(s \mid z^a_{12})]$ | 0.446583 | 0.361816 | 0.457171 | 0.0349602 | 0.146052 |
| $E[\Pr(s \mid z^a_{34})]$ | 0.253417 | 0.140924 | 0.178064 | 0.131529 | 0.549484 |

Probabilities for initial state 4 given accounting information $z^a \left(b = \frac{1}{2}\right)$

These expected probabilities are combined with state prices to determine expected logarithmic returns to a Kelly strategy conditional on the initial state $i$ and the signal $z$.

$$E[r_i \mid z] = E[\Pr(s \mid z)] \log(\Omega_i E[\Pr(s \mid z)])$$

Weighting the expectations by the likelihood of the signal and the initial state

$(p_{ss})$ gives the unconditional expected logarithmic return.

$$E[r_i(z)] = \Pr(z)^T \begin{bmatrix} E[r_i \mid \emptyset] \\ E[r_i \mid z_1] \\ E[r_i \mid z_2] \\ E[r_i \mid z_3] \\ E[r_i \mid z_4] \end{bmatrix}$$

$$E[r(z)] = p_{ss}^T \begin{bmatrix} E[r_1(z)] \\ E[r_2(z)] \\ E[r_3(z)] \\ E[r_4(z)] \end{bmatrix}$$

for nonaccounting information and

$$E[r_i(z^a)] = \Pr(z^a)^T \begin{bmatrix} E[r_i \mid \emptyset] \\ E[r_i \mid z_{12}^a] \\ E[r_i \mid z_{34}^a] \end{bmatrix}$$

$$E[r(z^a)] = p_{ss}^T \begin{bmatrix} E[r_1(z^a)] \\ E[r_2(z^a)] \\ E[r_3(z^a)] \\ E[r_4(z^a)] \end{bmatrix}$$

for accounting information. Then, the expected gain is this unconditional return less $\log \frac{1}{\delta}$ by the Kelly-Ross theorem.

$$E[gain \mid z] = E[r(z)] - \log \frac{1}{\delta}$$

and

$$E[gain \mid z^a] = E[r(z^a)] - \log \frac{1}{\delta}$$

The expected gain equals the above reported mutual information.

$$E\left[gain \mid z\left(b = \frac{1}{4}\right)\right] = I\left(s; z\left(b = \frac{1}{4}\right)\right) = 0.0484224$$

and

$$E\left[gain \mid z^a\left(b = \frac{1}{3}\right)\right] = I\left(s; z^a\left(b = \frac{1}{3}\right)\right) = 0.0388264$$

or

$$E\left[gain \mid z^a\left(b = \frac{1}{2}\right)\right] = I\left(s; z^a\left(b = \frac{1}{2}\right)\right) = 0.0890673$$

## 3.2 Alternative likelihood functions

Now, suppose the likelihood function associated with noisy other (nonaccounting) information is

$$\ell(p; z) \propto (1)^{\Im(\emptyset)} \left(p_1^b\right)^{\Im(z_1)} \times \cdots \times \left(p_4^b\right)^{\Im(z_4)}$$

Then, the posterior distribution $\Pr(p \mid z) \propto \Pr(p)\,\ell(p;z)$ is Dirichlet. Even for $b=1$, the likelihood functions of this section (unlike those of the previous section) do not perfectly indicate the transition state. In other words, the prior distribution continues to influence (posterior) beliefs.

Further, suppose the likelihood function associated with imperfect, coarse accounting information is[7]

$$\ell(p;z^a) \propto (1)^{\Im(\emptyset)} (p_1+p_2)^{\Im(z_{12}^a)} (p_3+p_4)^{\Im(z_{34}^a)}$$

The posterior distribution, $\Pr(p \mid z^a) \propto \Pr(p)\,\ell(p;z^a)$, is no longer Dirichlet but rather hyper- (or generalized) Dirichlet. First, we analytically normalize the posterior.

$$\begin{aligned}
&\Pr(p \mid z^a) \\
=\;& \left(\frac{\Gamma(a_1+a_2+a_3+a_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} p_1^{a_1-1} p_2^{a_2-1} p_3^{a_3-1} (1-p_1-p_2-p_3)^{a_4-1}\right)^{\Im(\emptyset)} \\
& \left(\begin{array}{c} \frac{\Gamma(1+a_1+a_2+a_3+a_4)}{(a_1+a_2)\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} \\ \times p_1^{a_1-1} p_2^{a_2-1} p_3^{a_3-1} (1-p_1-p_2-p_3)^{a_4-1} (p_1+p_2) \end{array}\right)^{\Im(z_{12}^a)} \\
& \left(\begin{array}{c} \frac{\Gamma(1+a_1+a_2+a_3+a_4)}{(a_3+a_4)\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} \\ \times p_1^{a_1-1} p_2^{a_2-1} p_3^{a_3-1} (1-p_1-p_2-p_3)^{a_4-1} (p_3+p_4) \end{array}\right)^{\Im(z_{34}^a)}
\end{aligned}$$

with expected value

$$\begin{aligned}
&E[p \mid z^a] \\
=\;& \left(\begin{array}{c} \frac{a_1}{a_1+a_2+a_3+a_4}, \frac{a_2}{a_1+a_2+a_3+a_4}, \\ \frac{a_3}{a_1+a_2+a_3+a_4}, \frac{a_4}{a_1+a_2+a_3+a_4} \end{array}\right)^{\Im(\emptyset)} \\
& \left(\begin{array}{c} \frac{a_1(1+a_1+a_2)}{(a_1+a_2)(1+a_1+a_2+a_3+a_4)}, \frac{a_2(1+a_1+a_2)}{(a_1+a_2)(1+a_1+a_2+a_3+a_4)}, \\ \frac{a_3}{1+a_1+a_2+a_3+a_4}, \frac{a_4}{1+a_1+a_2+a_3+a_4} \end{array}\right)^{\Im(z_{12}^a)} \\
& \left(\begin{array}{c} \frac{a_1}{1+a_1+a_2+a_3+a_4}, \frac{a_2}{1+a_1+a_2+a_3+a_4}, \\ \frac{a_3(1+a_3+a_4)}{(a_3+a4)(1+a_1+a_2+a_3+a_4)}, \frac{a_4(1+a_3+a_4)}{(a_3+a4)(1+a_1+a_2+a_3+a_4)} \end{array}\right)^{\Im(z_{34}^a)}
\end{aligned}$$

Since analytical normalization can be problematic we could also address this problem by generating posterior probabilities and means of the posterior probabilities via Markov chain Monte Carlo (McMC) simulation, specifically via Metropolis-Hastings algorithm (see the appendix for details). We don't report results for McMC simulation as they are (roughly) the same as those reported.[8]

[7] $(p_3+p_4) = (1-p_1-p_2)$ but is written in the former form to ease interpretation.

[8] Precision of the (mean) results depends on the number of draws as the variance of the mean is decreasing in the number of draws.

Finally, suppose we observe both coarse accounting and noisy nonaccounting information. The accounting information serves to highlight a subset of transition states and the nonaccounting information reinforces the likelihood of the remaining transition states. The likelihood function is

$$\begin{aligned} \ell\left(p; z, z^{a}\right) \quad \propto \quad & (1)^{\Im(\emptyset)} \left(p_1^b \left(p_1 + p_2\right)\right)^{\Im\left(z_1, z_{12}^a\right)} \left(\left(p_1 + p_2\right) p_2^b\right)^{\Im\left(z_2, z_{12}^a\right)} \\ & \times \left(p_3^b \left(p_3 + p_4\right)\right)^{\Im\left(z_3, z_{34}^a\right)} \left(\left(p_3 + p_4\right) p_4^b\right)^{\Im\left(z_4, z_{34}^a\right)} \end{aligned}$$

Again, the posterior distribution is hyper-Dirichlet.

$$\begin{aligned} & \Pr\left(p \mid z, z^{a}\right) \\ = \quad & \left(\frac{\Gamma\left(a_1 + a_2 + a_3 + a_4\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)} p_1^{a_1 - 1} p_2^{a_2 - 1} p_3^{a_3 - 1} \left(1 - p_1 - p_2 - p_3\right)^{a_4 - 1}\right)^{\Im(\emptyset)} \\ & \left(\begin{array}{c} \frac{\Gamma(1 + a_1 + a_2 + a_3 + a_4 + b)}{(a_1 + a_2 + b)\Gamma(a_1 + b)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} \\ \times p_1^{a_1 + b - 1} p_2^{a_2 - 1} p_3^{a_3 - 1} \left(1 - p_1 - p_2 - p_3\right)^{a_4 - 1} \left(p_1 + p_2\right) \end{array}\right)^{\Im\left(z_1, z_{12}^a\right)} \\ & \left(\begin{array}{c} \frac{\Gamma(1 + a_1 + a_2 + a_3 + a_4 + b)}{(a_1 + a_2 + b)\Gamma(a_1)\Gamma(a_2 + b)\Gamma(a_3)\Gamma(a_4)} \\ \times p_1^{a_1 - 1} p_2^{a_2 + b - 1} p_3^{a_3 - 1} \left(1 - p_1 - p_2 - p_3\right)^{a_4 - 1} \left(p_1 + p_2\right) \end{array}\right)^{\Im\left(z_2, z_{12}^a\right)} \\ & \left(\begin{array}{c} \frac{\Gamma(1 + a_1 + a_2 + a_3 + a_4 + b)}{(a_3 + a_4 + b)\Gamma(a_1)\Gamma(a_2)\Gamma(a_3 + b)\Gamma(a_4)} \\ \times p_1^{a_1 - 1} p_2^{a_2 - 1} p_3^{a_3 + b - 1} \left(1 - p_1 - p_2 - p_3\right)^{a_4 - 1} \left(p_3 + p_4\right) \end{array}\right)^{\Im\left(z_3, z_{34}^a\right)} \\ & \left(\begin{array}{c} \frac{\Gamma(1 + a_1 + a_2 + a_3 + a_4 + b)}{(a_3 + a_4 + b)\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4 + b)} \\ \times p_1^{a_1 - 1} p_2^{a_2 - 1} p_3^{a_3 - 1} \left(1 - p_1 - p_2 - p_3\right)^{a_4 + b - 1} \left(p_3 + p_4\right) \end{array}\right)^{\Im\left(z_3, z_{34}^a\right)} \end{aligned}$$

with expected value

$$
\begin{aligned}
&\Pr\left[p \mid z, z^a\right] \\
= \quad &\begin{pmatrix} \frac{a_1}{a_1+a_2+a_3+a_4}, \frac{a_2}{a_1+a_2+a_3+a_4}, \\ \frac{a_3}{a_1+a_2+a_3+a_4}, \frac{a_4}{a_1+a_2+a_3+a_4} \end{pmatrix}^{\Im(\emptyset)} \\
&\begin{pmatrix} \frac{(a_1+b)(1+a_1+a_2+b)}{(a_1+a_2+b)(1+a_1+a_2+a_3+a_4+b)}, \frac{a_2(1+a_1+a_2+b)}{(a_1+a_2+b)(1+a_1+a_2+a_3+a_4+b)}, \\ \frac{a_3}{1+a_1+a_2+a_3+a_{4+b}}, \frac{a_4}{1+a_1+a_2+a_3+a_4+b} \end{pmatrix}^{\Im(z_1, z_{12}^a)} \\
&\begin{pmatrix} \frac{a_1(1+a_1+a_2+b)}{(a_1+a_2+b)(1+a_1+a_2+a_3+a_4+b)}, \frac{(a_2+b)(1+a_1+a_2+b)}{(a_1+a_2+b)(1+a_1+a_2+a_3+a_4+b)}, \\ \frac{a_3}{1+a_1+a_2+a_3+a_{4+b}}, \frac{a_4}{1+a_1+a_2+a_3+a_4+b} \end{pmatrix}^{\Im(z_2, z_{12}^a)} \\
&\begin{pmatrix} \frac{a_1}{1+a_1+a_2+a_3+a_{4+b}}, \frac{a_2}{1+a_1+a_2+a_3+a_4+b}, \\ \frac{(a_3+b)(1+a_3+a_4+b)}{(a_3+a_4+b)(1+a_1+a_2+a_3+a_4+b)}, \frac{a_4(1+a_3+a_4+b)}{(a_3+a_4+b)(1+a_1+a_2+a_3+a_4+b)} \end{pmatrix}^{\Im(z_3, z_{34}^a)} \\
&\begin{pmatrix} \frac{a_1}{1+a_1+a_2+a_3+a_{4+b}}, \frac{a_2}{1+a_1+a_2+a_3+a_4+b}, \\ \frac{a_3(1+a_3+a_4+b)}{(a_3+a_4+b)(1+a_1+a_2+a_3+a_4+b)}, \frac{(a_4+b)(1+a_3+a_4+b)}{(a_3+a_4+b)(1+a_1+a_2+a_3+a_4+b)} \end{pmatrix}^{\Im(z_3, z_{34}^a)}
\end{aligned}
$$

Probability means readily translate into expected gains from alternative information sources via mutual information for a Kelly investment strategy as in the previous section.

### 3.2.1 Expected gain from information

We experiment with $b = \frac{1}{3}$, $b = \frac{1}{2}$, and $b = \frac{2}{3}$ for other information and compare expected gains with coarse accounting information as well as a combination of accounting and other information. Expected gains from information when employing a Kelly investment strategy are again reflected in mutual information, $I(s; z)$.

| | $z\left(b = \frac{1}{3}\right)$ | $z\left(b = \frac{1}{2}\right)$ | $z\left(b = \frac{2}{3}\right)$ | $z^a$ | $z\left(b = \frac{1}{3}\right), z^a$ |
|---|---|---|---|---|---|
| $I(s; z)$ | 0.0484224 | 0.0829467 | 0.11687 | 0.0890673 | 0.151037 |

Noisy other information with noise parameter $b = \frac{2}{3}$ offers greater benefits than coarse accounting information. However, coarse accounting information combined with noisy other information with noise parameter $b = \frac{1}{3}$ offers significantly greater expected benefits than other information with noise parameter $b = \frac{2}{3}$ indicating the complementary nature of the two information sources.

Details are similar to those of the previous section and include posterior probabilities associated with five signals over the four transition-states for each of four initial states or 80 joint probabilities and probability means for other (nonaccounting) information and for accounting combined with other information. Accounting information involves three signals over the four transition-states for each of four initial states or 48 joint probabilities and probability means. Again, probability means are the key to implementing a Kelly investment strategy. Posterior mean probabilities are tabulated below by initial state for accounting and other information.

Initial state 1 involves

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.40373 | 0.0486327 | 0.502097 | 0.0455403 |
| $E[\Pr(s \mid z_1)]$ | 0.282611 | 0.552797 | 0.0364746 | 0.376573 | 0.0341552 |
| $E[\Pr(s \mid z_2)]$ | 0.0340429 | 0.302797 | 0.286475 | 0.376573 | 0.0341552 |
| $E[\Pr(s \mid z_3)]$ | 0.351468 | 0.302797 | 0.0364746 | 0.626573 | 0.0341552 |
| $E[\Pr(s \mid z_4)]$ | 0.0318782 | 0.302797 | 0.0364746 | 0.376573 | 0.284155 |

Probabilities for initial state 1 given other information $z\left(b=\frac{1}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.40373 | 0.0486327 | 0.502097 | 0.0455403 |
| $E[\Pr(s \mid z_1)]$ | 0.282611 | 0.602487 | 0.0324218 | 0.334731 | 0.0303602 |
| $E[\Pr(s \mid z_2)]$ | 0.0340429 | 0.269153 | 0.365755 | 0.334731 | 0.0303602 |
| $E[\Pr(s \mid z_3)]$ | 0.351468 | 0.269153 | 0.0324218 | 0.668065 | 0.0303602 |
| $E[\Pr(s \mid z_4)]$ | 0.0318782 | 0.269153 | 0.0324218 | 0.334731 | 0.363694 |

Probabilities for initial state 1 given other information $z\left(b=\frac{1}{2}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.40373 | 0.0486327 | 0.502097 | 0.0455403 |
| $E[\Pr(s \mid z_1)]$ | 0.282611 | 0.642238 | 0.0291796 | 0.301258 | 0.0273242 |
| $E[\Pr(s \mid z_2)]$ | 0.0340429 | 0.242238 | 0.42918 | 0.301258 | 0.0273242 |
| $E[\Pr(s \mid z_3)]$ | 0.351468 | 0.242238 | 0.0291796 | 0.701258 | 0.0273242 |
| $E[\Pr(s \mid z_4)]$ | 0.0318782 | 0.242238 | 0.0291796 | 0.301258 | 0.427324 |

Probabilities for initial state 1 given other information $z\left(b=\frac{2}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.40373 | 0.0486327 | 0.502097 | 0.0455403 |
| $E[\Pr(s \mid z_{12}^a)]$ | 0.316654 | 0.648111 | 0.0780705 | 0.251049 | 0.0227701 |
| $E[\Pr(s \mid z_{34}^a)]$ | 0.383346 | 0.201865 | 0.0243164 | 0.70947 | 0.064349 |

Probabilities for initial state 1 given accounting information $z^a$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.40373 | 0.0486327 | 0.502097 | 0.0455403 |
| $E[\Pr(s \mid z_1, z_{12}^a)]$ | 0.282611 | 0.717928 | 0.0473702 | 0.215184 | 0.0195173 |
| $E[\Pr(s \mid z_2, z_{12}^a)]$ | 0.0340429 | 0.393249 | 0.37205 | 0.215184 | 0.0195173 |
| $E[\Pr(s \mid z_3, z_{34}^a)]$ | 0.351468 | 0.173027 | 0.0208426 | 0.764459 | 0.0416715 |
| $E[\Pr(s \mid z_4, z_{34}^a)]$ | 0.0318782 | 0.173027 | 0.0208426 | 0.459443 | 0.346687 |

Probabilities for initial state 1 given other and accounting information $z\left(b=\frac{1}{3}\right), z^a$

Initial state 2 involves

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.326953 | 0.30984 | 0.310455 | 0.0527523 |
| $E[\Pr(s \mid z_1)]$ | 0.228867 | 0.495215 | 0.23238 | 0.232841 | 0.0395642 |
| $E[\Pr(s \mid z_2)]$ | 0.216888 | 0.245215 | 0.48238 | 0.232841 | 0.0395642 |
| $E[\Pr(s \mid z_3)]$ | 0.217319 | 0.245215 | 0.23238 | 0.482841 | 0.0395642 |
| $E[\Pr(s \mid z_4)]$ | 0.0369266 | 0.245215 | 0.23238 | 0.232841 | 0.289564 |

Probabilities for initial state 2 given other information $z\left(b=\frac{1}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.326953 | 0.30984 | 0.310455 | 0.0527523 |
| $E[\Pr(s \mid z_1)]$ | 0.228867 | 0.551302 | 0.20656 | 0.20697 | 0.0351682 |
| $E[\Pr(s \mid z_2)]$ | 0.216888 | 0.217969 | 0.539893 | 0.20697 | 0.0351682 |
| $E[\Pr(s \mid z_3)]$ | 0.217319 | 0.217969 | 0.20656 | 0.540303 | 0.0351682 |
| $E[\Pr(s \mid z_4)]$ | 0.0369266 | 0.217969 | 0.20656 | 0.20697 | 0.368502 |

Probabilities for initial state 2 given other information $z\left(b=\frac{1}{2}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.326953 | 0.30984 | 0.310455 | 0.0527523 |
| $E[\Pr(s \mid z_1)]$ | 0.228867 | 0.596172 | 0.185904 | 0.186273 | 0.0316514 |
| $E[\Pr(s \mid z_2)]$ | 0.216888 | 0.196172 | 0.585904 | 0.186273 | 0.0316514 |
| $E[\Pr(s \mid z_3)]$ | 0.217319 | 0.196172 | 0.185904 | 0.586273 | 0.0316514 |
| $E[\Pr(s \mid z_4)]$ | 0.0369266 | 0.196172 | 0.185904 | 0.186273 | 0.431651 |

Probabilities for initial state 2 given other information $z\left(b=\frac{2}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.326953 | 0.30984 | 0.310455 | 0.0527523 |
| $E[\Pr(s \mid z_{12}^a)]$ | 0.445755 | 0.420195 | 0.398201 | 0.155228 | 0.0263761 |
| $E[\Pr(s \mid z_{34}^a)]$ | 0.254245 | 0.163476 | 0.15492 | 0.582607 | 0.0989962 |

Probabilities for initial state 2 given accounting information $z^a$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.326953 | 0.30984 | 0.310455 | 0.0527523 |
| $E[\Pr(s \mid z_1, z^a_{12})]$ | 0.228867 | 0.574674 | 0.269666 | 0.133052 | 0.0226081 |
| $E[\Pr(s \mid z_2, z^a_{12})]$ | 0.216888 | 0.28456 | 0.559779 | 0.133052 | 0.0226081 |
| $E[\Pr(s \mid z_3, z^a_{34})]$ | 0.217319 | 0.140123 | 0.132788 | 0.672023 | 0.0550658 |
| $E[\Pr(s \mid z_4, z^a_{34})]$ | 0.0369266 | 0.140123 | 0.132788 | 0.324071 | 0.403018 |

Probabilities for initial state 2 given other and accounting information $z\left(b = \frac{1}{3}\right), z^a$

Initial state 3 involves

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.330195 | 0.323051 | 0.319998 | 0.0267554 |
| $E[\Pr(s \mid z_1)]$ | 0.231137 | 0.497646 | 0.242288 | 0.239999 | 0.0200666 |
| $E[\Pr(s \mid z_2)]$ | 0.226136 | 0.247646 | 0.492288 | 0.239999 | 0.0200666 |
| $E[\Pr(s \mid z_3)]$ | 0.223999 | 0.247646 | 0.242288 | 0.489999 | 0.0200666 |
| $E[\Pr(s \mid z_4)]$ | 0.0187288 | 0.247646 | 0.242288 | 0.239999 | 0.270067 |

Probabilities for initial state 3 given other information $z\left(b = \frac{1}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.330195 | 0.323051 | 0.319998 | 0.0267554 |
| $E[\Pr(s \mid z_1)]$ | 0.231137 | 0.553463 | 0.215367 | 0.213332 | 0.0178369 |
| $E[\Pr(s \mid z_2)]$ | 0.226136 | 0.22013 | 0.548701 | 0.213332 | 0.0178369 |
| $E[\Pr(s \mid z_3)]$ | 0.223999 | 0.22013 | 0.215367 | 0.546666 | 0.0178369 |
| $E[\Pr(s \mid z_4)]$ | 0.0187288 | 0.22013 | 0.215367 | 0.213332 | 0.35117 |

Probabilities for initial state 3 given other information $z\left(b = \frac{1}{2}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.330195 | 0.323051 | 0.319998 | 0.0267554 |
| $E[\Pr(s \mid z_1)]$ | 0.231137 | 0.598117 | 0.193831 | 0.191999 | 0.0160532 |
| $E[\Pr(s \mid z_2)]$ | 0.226136 | 0.198117 | 0.593831 | 0.191999 | 0.0160532 |
| $E[\Pr(s \mid z_3)]$ | 0.223999 | 0.198117 | 0.193831 | 0.591999 | 0.0160532 |
| $E[\Pr(s \mid z_4)]$ | 0.0187288 | 0.198117 | 0.193831 | 0.191999 | 0.416053 |

Probabilities for initial state 3 given other information $z\left(b = \frac{2}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.330195 | 0.323051 | 0.319998 | 0.0267554 |
| $E[\Pr(s \mid z^a_{12})]$ | 0.457272 | 0.417832 | 0.408791 | 0.159999 | 0.0133777 |
| $E[\Pr(s \mid z^a_{34})]$ | 0.242728 | 0.165098 | 0.161525 | 0.621419 | 0.0519575 |

Probabilities for initial state 3 given accounting information $z^a$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.330195 | 0.323051 | 0.319998 | 0.0267554 |
| $E[\Pr(s \mid z_1, z_{12}^a)]$ | 0.231137 | 0.572607 | 0.278784 | 0.137142 | 0.0114666 |
| $E[\Pr(s \mid z_2, z_{12}^a)]$ | 0.226136 | 0.284949 | 0.566442 | 0.137142 | 0.0114666 |
| $E[\Pr(s \mid z_3, z_{34}^a)]$ | 0.223999 | 0.141512 | 0.13845 | 0.69171 | 0.0283271 |
| $E[\Pr(s \mid z_4, z_{34}^a)]$ | 0.0187288 | 0.141512 | 0.13845 | 0.338796 | 0.381241 |

Probabilities for initial state 3 given other and accounting information $z\left(b=\frac{1}{3}\right), z^a$

Initial state 4 involves

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.281848 | 0.356128 | 0.0699203 | 0.292104 |
| $E[\Pr(s \mid z_1)]$ | 0.197294 | 0.461386 | 0.267096 | 0.0524403 | 0.219078 |
| $E[\Pr(s \mid z_2)]$ | 0.249289 | 0.211386 | 0.517096 | 0.0524403 | 0.219078 |
| $E[\Pr(s \mid z_3)]$ | 0.0489442 | 0.211386 | 0.267096 | 0.30244 | 0.219078 |
| $E[\Pr(s \mid z_4)]$ | 0.204473 | 0.211386 | 0.267096 | 0.0524403 | 0.469078 |

Probabilities for initial state 4 given other information $z\left(b=\frac{1}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.281848 | 0.356128 | 0.0699203 | 0.292104 |
| $E[\Pr(s \mid z_1)]$ | 0.197294 | 0.521232 | 0.237418 | 0.0466136 | 0.194736 |
| $E[\Pr(s \mid z_2)]$ | 0.249289 | 0.187899 | 0.570752 | 0.0466136 | 0.194736 |
| $E[\Pr(s \mid z_3)]$ | 0.0489442 | 0.187899 | 0.237418 | 0.379947 | 0.194736 |
| $E[\Pr(s \mid z_4)]$ | 0.204473 | 0.187899 | 0.237418 | 0.0466136 | 0.528069 |

Probabilities for initial state 4 given other information $z\left(b=\frac{1}{2}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.281848 | 0.356128 | 0.0699203 | 0.292104 |
| $E[\Pr(s \mid z_1)]$ | 0.197294 | 0.569109 | 0.213677 | 0.0419522 | 0.175262 |
| $E[\Pr(s \mid z_2)]$ | 0.249289 | 0.169109 | 0.613677 | 0.0419522 | 0.175262 |
| $E[\Pr(s \mid z_3)]$ | 0.0489442 | 0.169109 | 0.213677 | 0.441952 | 0.175262 |
| $E[\Pr(s \mid z_4)]$ | 0.204473 | 0.169109 | 0.213677 | 0.0419522 | 0.575262 |

Probabilities for initial state 4 given other information $z\left(b=\frac{2}{3}\right)$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.281848 | 0.356128 | 0.0699203 | 0.292104 |
| $E[\Pr(s \mid z_{12}^a)]$ | 0.446583 | 0.361816 | 0.457171 | 0.0349602 | 0.146052 |
| $E[\Pr(s \mid z_{34}^a)]$ | 0.253417 | 0.140924 | 0.178064 | 0.131529 | 0.549484 |

Probabilities for initial state 4 given accounting information $z^a$

| | $\Pr(signal)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| $E[\Pr(s \mid \emptyset)]$ | 0.3 | 0.281848 | 0.356128 | 0.0699203 | 0.292104 |
| $E[\Pr(s \mid z_1, z_{12}^a)]$ | 0.197294 | 0.535086 | 0.309761 | 0.0299659 | 0.125187 |
| $E[\Pr(s \mid z_2, z_{12}^a)]$ | 0.249289 | 0.245152 | 0.599695 | 0.0299659 | 0.125187 |
| $E[\Pr(s \mid z_3, z_{34}^a)]$ | 0.0489442 | 0.120792 | 0.152626 | 0.421361 | 0.305221 |
| $E[\Pr(s \mid z_4, z_{34}^a)]$ | 0.204473 | 0.120792 | 0.152626 | 0.07306 | 0.653522 |

Probabilities for initial state 4 given other and accounting information $z\left(b = \frac{1}{3}\right), z^a$

# Appendix

Metropolis-Hastings draws from either posterior distribution involving coarse accounting information utilize independent multivariate normal draws tempered by $\alpha$-probability that previous draws are replaced by the proposal. Details are as follows.

1. For $d$ states, start with a vector of proposed probabilities, say $\theta^0 = \frac{1}{d}\iota$.

2. Draw $d$ independent values from a zero mean normal density with variance $\sigma I$ and label the vector $\eta^1$ (often $\sigma = \frac{1}{d}$, $\sigma$ is regarded as the tuning parameter and may require adjustment to adequately and efficiently cover the parameter space).

3. Create a trial value of $\theta^1 = \theta^0 + \eta^1 - \overline{\eta^1}\iota$ where $\overline{\eta^1}$ is the sample mean of $\eta^1$.

4. Draw a uniform$(0,1)$ variable $\mu_1$.

5. Typically, the next step calculates the ratio $\alpha = \frac{L\left(\theta^1 |, z_a, z\right)\phi\left(\theta^1\right)}{L\left(\theta^0 |, z_a, z\right)\phi\left(\theta^0\right)}$ where $L\left(\theta^1 |, z_a, z\right)$ is the likelihood for the proposal and $\phi\left(\theta^1\right)$ is the prior for the proposal with $L\left(\theta^0 |, z_a, z\right)$ and $\phi\left(\theta^0\right)$ the analogs for the initial or previous draw. However, the numerator and denominator (posterior kernel) values can become extremely small for large quantities of evidence. Therefore, the ratio is modified as $\alpha = \frac{\log\left[L\left(\theta^1 |, z_a, z\right)\phi\left(\theta^1\right)\right] - \log\left[L\left(\theta^0 |, z_a, z\right)\phi\left(\theta^0\right)\right]}{1}$.

6. If $\mu_1 \leq \alpha$, accept $\theta^1$; otherwise, reject $\theta^1$ and let $\theta^1 = \theta^0$.

7. Repeat the process many times, adjusting the tuning parameters if necessary. For sufficiently large $t$, $\theta^t$ is a draw from the marginal posterior.

We find that rescaling parameters by 500 (to reduce the variance) and simulating at least $10,000$ draws with a burn-in of $1,000$ produces results reasonably close to our analytical solution for probability means.