Information Synergy, part 2:
Accounting and other information

Discussion of information synergy is extended to consider the impact of accounting and other information on Kelly-Ross investment strategies. This additional information may inform economic agent's about the initial state, s_0, the post-transition state, s_t, or both.

Accounting information may be coarse, late, or both. That is, coarse accounting information, z_a, does not distinguish each state in the following sense.

$$\Pr\left(z_a = 1 \mid s_0, s_t\right) = \Pr\left(z_a = 2 \mid s_0, s_t\right), \quad for\ z_a = 12$$

and

$$\Pr\left(z_a = 3 \mid s_0, s_t\right) = \Pr\left(z_a = 4 \mid s_0, s_t\right), \quad for\ z_a = 34$$

While late accounting information reveals the state but with a time lag.

On the other hand, other information, z, is both finer and more timely but may be biased or misinterpreted in such manner to create bias.

1 Initial state information

If initial state information perfectly reveals the initial state, we know from part 1 the expected logarithmic return is $\log \frac{1}{\delta}$ and the expected gain from the initial state information relative to ignoring initial state information is given by mutual information, $I\left(s_t; z = s_0\right) = H\left(s_t\right) + H\left(s_0\right) - H\left(s_t, s_0\right)$. Now, suppose (price dynamic) initial state information is ignored and accounting information imperfectly informs economic agents about the initial state. The expected gain from the accounting again is given by mutual information, $I\left(s_t; z_a\right) = H\left(s_t\right) + H\left(z_a\right) - H\left(s_t, z_a\right)$. An example helps clarify.

Example 1 (Accounting supplies initial state information) *Suppose there are four states and four assets where initial state one involves the following returns matrix, A_1, and normalized price vector, x.*

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1.10 & 1.05 & \frac{1}{1.10} & 1.10 \\ \frac{1}{1.10} & 1.10 & 1.05 & 1.10 \\ 1.10 & 1.10 & \frac{1}{1.10} & 1.05 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The associated state price vector solves $A_1 y_1 = x$ to yield

$$y_1^T = \begin{bmatrix} 0.3900220 & 0.0495764 & 0.5108252 & 0.0495764 \end{bmatrix}$$

Initial state two involves the following returns matrix, A_2, and normalized price vector, x.

$$A_2 = \begin{bmatrix} 1.00 & 1.25 & \frac{1}{1.10} & 1.00 \\ 1.05 & 1.05 & 1.05 & 1.05 \\ \frac{1}{1.10} & 1.00 & 1.25 & 1.00 \\ 1.25 & 1.00 & \frac{1}{1.10} & 1.00 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The associated state price vector is

$$y_2^T = \begin{bmatrix} 0.29931973 & 0.29931973 & 0.29931973 & 0.05442177 \end{bmatrix}$$

Initial state three involves the following returns matrix, A_3, and normalized price vector, x.

$$A_3 = \begin{bmatrix} 1.00 & 1.35 & 0.80 & 1.00 \\ 0.80 & 1.00 & 1.35 & 1.00 \\ 1.05 & 1.05 & 1.05 & 1.05 \\ 1.25 & 1.00 & \frac{1}{1.10} & 1.00 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The associated state price vector is

$$y_3^T = \begin{bmatrix} 0.30288837 & 0.31270213 & 0.30913349 & 0.02765696 \end{bmatrix}$$

Initial state four involves the following returns matrix, A_4, and normalized price vector, x.

$$A_4 = \begin{bmatrix} 1.00 & 1.30 & \frac{1}{1.10} & 1.00 \\ \frac{1}{1.10} & 1.00 & 1.00 & 1.40 \\ 1.40 & 1.00 & \frac{1}{1.10} & 1.00 \\ 1.10 & 1.10 & 1.10 & 1.10 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The associated state price vector is

$$y_4^T = \begin{bmatrix} 0.24161950 & 0.32215934 & 0.06312582 & 0.28218625 \end{bmatrix}$$

Then, the transition-state price matrix is

$$P = \begin{bmatrix} 0.3900220 & 0.0495764 & 0.5108252 & 0.0495764 \\ 0.29931973 & 0.29931973 & 0.29931973 & 0.05442177 \\ 0.30288837 & 0.31270213 & 0.30913349 & 0.02765696 \\ 0.24161950 & 0.32215934 & 0.06312582 & 0.28218625 \end{bmatrix}$$

with implied transition-probability matrix

$$
\begin{aligned}
F &= \Pr\left(s_t \mid s_0\right) = \frac{1}{\delta} D P D^{-1} \\
&= \frac{1}{0.96604689} \begin{bmatrix} 1.893980 & 0 & 0 & 0 \\ 0 & 1.998589 & 0 & 0 \\ 0 & 0 & 1.994628 & 0 \\ 0 & 0 & 0 & 2.134306 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 0.3900220 & 0.0495764 & 0.5108252 & 0.0495764 \\ 0.29931973 & 0.29931973 & 0.29931973 & 0.05442177 \\ 0.30288837 & 0.31270213 & 0.30913349 & 0.02765696 \\ 0.24161950 & 0.32215934 & 0.06312582 & 0.28218625 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 0.5279886 & 0 & 0 & 0 \\ 0 & 0.5003530 & 0 & 0 \\ 0 & 0 & 0.5013467 & 0 \\ 0 & 0 & 0 & 0.4685365 \end{bmatrix} \\
&= \begin{bmatrix} 0.4037298 & 0.04863275 & 0.50209713 & 0.04554028 \\ 0.3269529 & 0.30983975 & 0.31045508 & 0.05275229 \\ 0.3301952 & 0.32305094 & 0.31999843 & 0.02675541 \\ 0.2818480 & 0.35612759 & 0.06992034 & 0.29210409 \end{bmatrix}
\end{aligned}
$$

and steady-state probabilities

$$p_{ss}^T = \begin{bmatrix} 0.35282884 & 0.22502190 & 0.36874554 & 0.05340373 \end{bmatrix}$$

where $p_{ss}^T F = p_{ss}^T$.

Accounting and other information inform the initial state through $\Pr\left(z_a, z \mid s_0, s_t\right)$

$\Pr\left(z_a, z \mid s_0, s_t\right)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.25	0.25	0.25	0.25
$s_0 = s_1, s_t = s_2$	0.25	0.25	0.25	0.25
$s_0 = s_1, s_t = s_3$	0.25	0.25	0.25	0.25
$s_0 = s_1, s_t = s_4$	0.25	0.25	0.25	0.25
$s_0 = s_2, s_t = s_1$	0.25	0.25	0.25	0.25
$s_0 = s_2, s_t = s_2$	0.25	0.25	0.25	0.25
$s_0 = s_2, s_t = s_3$	0.25	0.25	0.25	0.25
$s_0 = s_2, s_t = s_4$	0.25	0.25	0.25	0.25
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.0

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.25	0.25	0.25	0.25
$s_0 = s_3, s_t = s_2$	0.25	0.25	0.25	0.25
$s_0 = s_3, s_t = s_3$	0.25	0.25	0.25	0.25
$s_0 = s_3, s_t = s_4$	0.25	0.25	0.25	0.25
$s_0 = s_4, s_t = s_1$	0.25	0.25	0.25	0.25
$s_0 = s_4, s_t = s_2$	0.25	0.25	0.25	0.25
$s_0 = s_4, s_t = s_3$	0.25	0.25	0.25	0.25
$s_0 = s_4, s_t = s_4$	0.25	0.25	0.25	0.25

Hence, the joint distribution is $\Pr(z_a, z, s_0, s_t) = \Pr(z_a, z \mid s_0, s_t) \Pr(s_t \mid s_0) \Pr(s_0)$ *where* $\Pr(s_0) = p_{ss}$

$\Pr(z_a, z, s_0, s_t)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.035611883	0.035611883	0.035611883	0.035611883
$s_0 = s_1, s_t = s_2$	0.004289759	0.004289759	0.004289759	0.004289759
$s_0 = s_1, s_t = s_3$	0.044288586	0.044288586	0.044288586	0.044288586
$s_0 = s_1, s_t = s_4$	0.004016981	0.004016981	0.004016981	0.004016981
$s_0 = s_2, s_t = s_1$	0.018392889	0.018392889	0.018392889	0.018392889
$s_0 = s_2, s_t = s_2$	0.017430182	0.017430182	0.017430182	0.017430182
$s_0 = s_2, s_t = s_3$	0.017464798	0.017464798	0.017464798	0.017464798
$s_0 = s_2, s_t = s_4$	0.002967605	0.002967605	0.002967605	0.002967605
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.0

$\Pr(z_a, z, s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.030439503	0.030439503	0.030439503	0.030439503
$s_0 = s_3, s_t = s_2$	0.029780898	0.029780898	0.029780898	0.029780898
$s_0 = s_3, s_t = s_3$	0.029499498	0.029499498	0.029499498	0.029499498
$s_0 = s_3, s_t = s_4$	0.002466484	0.002466484	0.002466484	0.002466484
$s_0 = s_4, s_t = s_1$	0.003762933	0.003762933	0.003762933	0.003762933
$s_0 = s_4, s_t = s_2$	0.004754635	0.004754635	0.004754635	0.004754635
$s_0 = s_4, s_t = s_3$	0.000933502	0.000933502	0.000933502	0.000933502
$s_0 = s_4, s_t = s_4$	0.003899862	0.003899862	0.003899862	0.003899862

$\Pr(z_a, s_0) = \sum_{z,s_t} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z_a, s_0)$	$z_a = 12$	$z_a = 34$
$s_0 = 1$	0.352828837	0.0
$s_0 = 2$	0.225021899	0.0
$s_0 = 3$	0.0	0.368745534
$s_0 = 4$	0.0	0.053403731
$\Pr(z_a)$	0.577850736	0.422149264

$\Pr(z_a, s_t) = \sum_{z,s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z_a, s_t)$	$z_a = 12$	$z_a = 34$
$s_t = 1$	0.216019091	0.136809746
$s_t = 2$	0.086879765	0.138142133
$s_t = 3$	0.247013534	0.121731999
$s_t = 4$	0.027938344	0.025465386
$\Pr(z_a)$	0.577850736	0.422149264

$\Pr(s_0, s_t) = \sum_{z_a,z} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(s_0, s_t)$	$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$	$\Pr(s_0)$
$s_0 = 1$	0.142447533	0.017159037	0.177154343	0.016067924	0.352828837
$s_0 = 2$	0.073571558	0.069720729	0.069859192	0.01187042	0.225021899
$s_0 = 3$	0.121758013	0.119123591	0.117997992	0.009865938	0.368745534
$s_0 = 4$	0.015051733	0.019018542	0.003734008	0.015599448	0.053403731
$\Pr(s_t)$	0.352828837	0.225021899	0.368745534	0.053403731	

These data indicate when the initial state is known the long-run expected *logarithmic return is* $\log \frac{1}{\delta} = 0.03454291$ *and the expected gain from initial state information is*

$$\begin{aligned} I(s_t; s_0) &= H(s_t) + H(s_0) - H(s_t, s_0) \\ &= 1.227545054 + 1.227545054 - 2.368419731 \\ &= 0.086670377 \end{aligned}$$

If the (price dynamic) initial state information is ignored except through accounting information, the expected gain from the accounting information is

$$\begin{aligned} I(s_t; z_a) &= H(s_t) + H(z_a) - H(s_t, z_a) \\ &= 1.227545054 + 0.680976249 - 1.884065075 \\ &= 0.024456227 \end{aligned}$$

or an expected logarithmic return equal to -0.027671244 *while reliance on the other information,* z*, offers no advantage*

$$\begin{aligned} I(s_t; z) &= H(s_t) + H(z) - H(s_t, z) \\ &= 1.227545054 + 1.386294361 - 2.613839415 \\ &= 0.0 \end{aligned}$$

or an expected logarithmic return equal to -0.052127471.

For clarity, we present details associated with expected logarithmic returns. If the initial state information is utilized

$$\begin{aligned} & E[r \mid s_0] \\ = \; & p_{ss}^T \begin{bmatrix} E[r_1 \mid s_0] \\ E[r_2 \mid s_0] \\ E[r_3 \mid s_0] \\ E[r_4 \mid s_0] \end{bmatrix} \\ = \; & \begin{bmatrix} 0.35282884 & 0.22502190 & 0.36874554 & 0.05340373 \end{bmatrix} \begin{bmatrix} 0.000491095 \\ 0.049270279 \\ 0.049187539 \\ 0.096342699 \end{bmatrix} \\ = \; & 0.03454291 \end{aligned}$$

where

$$\begin{aligned} & E[r_1 \mid s_0] \\ = \; & F_1 \log\left(\Omega_1 F_1^T\right) \\ = \; & \begin{bmatrix} 0.4037298 & 0.04863275 & 0.50209713 & 0.04554028 \end{bmatrix} \\ & \times \log\left(\begin{bmatrix} \frac{1}{0.3900220} & 0 & 0 & 0 \\ 0 & \frac{1}{0.0495764} & 0 & 0 \\ 0 & 0 & \frac{1}{0.5108252} & 0 \\ 0 & 0 & 0 & \frac{1}{0.0495764} \end{bmatrix} \begin{bmatrix} 0.4037298 \\ 0.04863275 \\ 0.50209713 \\ 0.04554028 \end{bmatrix} \right) \\ = \; & 0.000491095 \end{aligned}$$

and Ω_j is a diagonal matrix with $\frac{1}{y_j}$ along the diagonal for initial state j. Details for other initial states are analogous.

On the other hand, the base case in which initial state information is ignored so that the same investment strategy prevails over each initial state yields

$$\begin{aligned}
& E[r \mid \emptyset] \\
= \; & p_{ss}^T \begin{bmatrix} E[r_1 \mid \emptyset] \\ E[r_2 \mid \emptyset] \\ E[r_3 \mid \emptyset] \\ E[r_4 \mid \emptyset] \end{bmatrix} \\
= \; & \begin{bmatrix} 0.35282884 & 0.22502190 & 0.36874554 & 0.05340373 \end{bmatrix} \begin{bmatrix} -0.127153106 \\ 0.029135433 \\ 0.018124603 \\ -0.383936895 \end{bmatrix} \\
= \; & -0.052127468
\end{aligned}$$

where

$$\begin{aligned}
& E[r_1 \mid \emptyset] \\
= \; & F_1 \log(\Omega_1 p_{ss}) \\
= \; & \begin{bmatrix} 0.4037298 & 0.04863275 & 0.50209713 & 0.04554028 \end{bmatrix} \\
& \times \log\left(\begin{bmatrix} \frac{1}{0.3900220} & 0 & 0 & 0 \\ 0 & \frac{1}{0.0495764} & 0 & 0 \\ 0 & 0 & \frac{1}{0.5108252} & 0 \\ 0 & 0 & 0 & \frac{1}{0.0495764} \end{bmatrix} \begin{bmatrix} 0.35282884 \\ 0.22502190 \\ 0.36874554 \\ 0.05340373 \end{bmatrix} \right) \\
= \; & -0.127153106
\end{aligned}$$

Again, details for other initial states are analogous.

If accounting information is utilized in place of initial state information, expected logarithmic returns are

$$\begin{aligned}
& E[r \mid z_a] \\
= \; & p_{ss}^T \begin{bmatrix} E[r_1 \mid z_a] \\ E[r_2 \mid z_a] \\ E[r_3 \mid z_a] \\ E[r_4 \mid z_a] \end{bmatrix} \\
= \; & \begin{bmatrix} 0.35282884 & 0.22502190 & 0.36874554 & 0.05340373 \end{bmatrix} \begin{bmatrix} -0.05374904 \\ -0.03626453 \\ 0.03561197 \\ -0.25613333 \end{bmatrix} \\
= \; & -0.027671244
\end{aligned}$$

where

$$\begin{aligned}
& E\left[r_1 \mid z_a\right] \\
= \; & \Pr\left(s_t \mid s_0 = 1, z_a = 12\right) \log\left(\Omega_1 \Pr\left(s_t \mid z_a = 12\right)\right) \\
& + \Pr\left(s_t \mid s_0 = 1, z_a = 34\right) \log\left(\Omega_1 \Pr\left(s_t \mid z_a = 34\right)\right) \\
= \; & \Pr\left(s_t \mid s_0 = 1, z_a = 12\right) \log\left(\Omega_1 \Pr\left(s_t \mid z_a = 12\right)\right) \\
& + 0 \log\left(\Omega_1 \Pr\left(s_t \mid z_a = 34\right)\right) \\
= \; & \begin{bmatrix} 0.4037298 & 0.04863275 & 0.50209713 & 0.04554028 \end{bmatrix} \\
& \times \log\left(\begin{bmatrix} \frac{1}{0.3900220} & 0 & 0 & 0 \\ 0 & \frac{1}{0.0495764} & 0 & 0 \\ 0 & 0 & \frac{1}{0.5108252} & 0 \\ 0 & 0 & 0 & \frac{1}{0.0495764} \end{bmatrix} \begin{bmatrix} 0.37383199 \\ 0.15034984 \\ 0.42746945 \\ 0.04834872 \end{bmatrix}\right) \\
= \; & -0.05374904
\end{aligned}$$

Details for other initial states are analogous.

If other information is utilized in place of initial state information, expected logarithmic returns are

$$\begin{aligned}
& E\left[r \mid z\right] \\
= \; & p_{ss}^T \begin{bmatrix} E\left[r_1 \mid z\right] \\ E\left[r_2 \mid z\right] \\ E\left[r_3 \mid z\right] \\ E\left[r_4 \mid z\right] \end{bmatrix} \\
= \; & \begin{bmatrix} 0.35282884 & 0.22502190 & 0.36874554 & 0.05340373 \end{bmatrix} \begin{bmatrix} -0.12715312 \\ 0.02913542 \\ 0.01812459 \\ -0.38393691 \end{bmatrix} \\
= \; & -0.05212748
\end{aligned}$$

where

$$\begin{aligned}
& E\left[r_1 \mid z\right] \\
= \; & \Pr\left(s_t \mid s_0 = 1, z = 1\right) \log\left(\Omega_1 \Pr\left(s_t \mid z = 1\right)\right) \\
& + \Pr\left(s_t \mid s_0 = 1, z = 2\right) \log\left(\Omega_1 \Pr\left(s_t \mid z = 2\right)\right) \\
& + \Pr\left(s_t \mid s_0 = 1, z = 3\right) \log\left(\Omega_1 \Pr\left(s_t \mid z = 3\right)\right) \\
& + \Pr\left(s_t \mid s_0 = 1, z = 4\right) \log\left(\Omega_1 \Pr\left(s_t \mid z = 4\right)\right) \\
= \; & \begin{bmatrix} 0.08820721 + 0.08820721 + 0.08820721 + 0.08820721 \\ 0.05625547 + 0.05625547 + 0.05625547 + 0.05625547 \\ 0.09218638 + 0.09218638 + 0.09218638 + 0.09218638 \\ 0.01335093 + 0.01335093 + 0.01335093 + 0.01335093 \end{bmatrix}^T \\
& \times \log\left(\begin{bmatrix} \frac{1}{0.3900220} & 0 & 0 & 0 \\ 0 & \frac{1}{0.0495764} & 0 & 0 \\ 0 & 0 & \frac{1}{0.5108252} & 0 \\ 0 & 0 & 0 & \frac{1}{0.0495764} \end{bmatrix} \begin{bmatrix} 0.35282884 \\ 0.22502190 \\ 0.36874554 \\ 0.05340373 \end{bmatrix}\right) \\
= \; & \begin{bmatrix} 0.4037298 & 0.04863275 & 0.50209713 & 0.04554028 \end{bmatrix} \\
& \times \log\left(\begin{bmatrix} \frac{1}{0.3900220} & 0 & 0 & 0 \\ 0 & \frac{1}{0.0495764} & 0 & 0 \\ 0 & 0 & \frac{1}{0.5108252} & 0 \\ 0 & 0 & 0 & \frac{1}{0.0495764} \end{bmatrix} \begin{bmatrix} 0.35282884 \\ 0.22502190 \\ 0.36874554 \\ 0.05340373 \end{bmatrix}\right) \\
= \; & -0.12715312
\end{aligned}$$

Details for other initial states are analogous.

Results for various information profiles are summarized below.

information	$I\left(s_t; \mathit{info}\right)$	$E\left[r \mid \mathit{info}\right]$
none	0	−0.052127471
s_0	0.086670377	0.034542906
z_a	0.024456227	−0.027671244
z	0.0	−0.052127471
z_a, z	0.024456227	−0.027671244
s_0, z_a	0.086670377	0.034542906
s_0, z	0.086670377	0.034542906
s_0, z_a, z	0.086670377	0.034542906

Clearly, ignoring initial state information is costly as the agent could generate *better results by investing in the riskless asset in each initial state*

$$
\begin{aligned}
& p_{ss}^T \log \begin{bmatrix} r_1^f \\ r_2^f \\ r_3^f \\ r_4^f \end{bmatrix} \\
= & \begin{bmatrix} 0.35282884 & 0.22502190 & 0.36874554 & 0.05340373 \end{bmatrix} \log \begin{bmatrix} 1.00 \\ 1.05 \\ 1.05 \\ 1.10 \end{bmatrix} \\
= & \ 0.03405993
\end{aligned}
$$

Further, this riskless investment strategy can be bested even if the initial state is unknown and the riskless asset is not identifiable in each initial state.[1]

$$
\begin{array}{cc}
\max\limits_{w} & \sum_{i=1}^{4} p_i^{ss} F_i \log\left(A_i^T w\right) \\
s.t. & w^T \iota = 1
\end{array}
$$

where argmax is

$$
w^T = \begin{bmatrix} 0.0810813 & 0.167701 & 0.329349 & 0.421868 \end{bmatrix}
$$

yielding an optimal expected logarithmic return equal to

$$
\sum_{i=1}^{4} p_i^{ss} F_i \log\left(A_i^T w\right) = 0.034102
$$

Ignoring the initial state information is a convenient benchmark for mutual information. For a Kelly investment strategy, $\log \frac{1}{\delta}$ is the expected logarithmic return given the initial state information. Mutual information for the initial state information is the difference between $\log \frac{1}{\delta}$ and expected logarithmic return for the benchmark of ignoring this initial state information. Further, as illustrated in example 1, ignoring the initial state information is the common benchmark for determining value of various information sources via mutual information calculus. Mutual information is much faster, simpler, and more compact then brute force expected logarithmic return calculus. Accordingly, future examples primarily rely on mutual information for exploring the implications of various information scenarios.

[1] This may seem at odds with the benchmark case described as ignoring initial state information. However, we're effectively solving for n weights when there are n^2 states. Here the n weights are weights on nominal assets whereas in the benchmark case the n weights are on the states or Arrow-Debreu portfolios. When the number of states and weights are equal (as is the case when initial state information is exploited) the two approaches (weights on states and weights on assets) are comparable but here they are not comparable.

Example 2 (other information reveals the initial state) *Everything remains as in example 1 except the information structure. Accounting and other information inform the initial state through* $\Pr(z_a, z \mid s_0, s_t)$

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	1.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	1.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	1.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	1.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	1.0	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	1.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	1.0	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	1.0	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.0

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.0	0.0	1.0	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	1.0	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	1.0	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	1.0	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	1.0
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	1.0
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	1.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	1.0

Hence, the joint distribution is $\Pr(z_a, z, s_0, s_t) = \Pr(z_a, z \mid s_0, s_t)\Pr(s_t \mid s_0)\Pr(s_0)$

where $\Pr(s_0) = p_{ss}$

$\Pr(z_a, z, s_0, s_t)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.142447533	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.017159037	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.177154343	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.016067924	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.073571558	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.069720729	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.069859192	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.01187042	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.0

$\Pr(z_a, z, s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.121758013	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.119123591	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.117997992	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.009865938	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.015051733
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.019018542
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.003734008
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.015599448

$\Pr(z, s_0) = \sum_{z_a, s_t} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z, s_0)$	$z = 1$	$z = 2$	$z = 3$	$z = 4$
$s_0 = 1$	0.352828837	0.0	0.0	0.0
$s_0 = 2$	0.0	0.225021899	0.0	0.0
$s_0 = 3$	0.0	0.0	0.368745534	0.0
$s_0 = 4$	0.0	0.0	0.0	0.053403731
$\Pr(z)$	0.352828837	0.225021899	0.368745534	0.053403731

$\Pr(z, s_t) = \sum_{z_a, s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z, s_t)$	$z = 1$	$z = 2$	$z = 3$	$z = 4$
$s_t = 1$	0.142447533	0.073571558	0.121758013	0.015051733
$s_t = 2$	0.017159037	0.069720729	0.119123591	0.019018542
$s_t = 3$	0.177154343	0.069859192	0.117997992	0.003734008
$s_t = 4$	0.016067924	0.01187042	0.009865938	0.015599448
$\Pr(z)$	0.352828837	0.225021899	0.368745534	0.053403731

$\Pr(s_0, s_t) = \sum_{z_a, z} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(s_0, s_t)$	$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$	$\Pr(s_0)$
$s_0 = 1$	0.142447533	0.017159037	0.177154343	0.016067924	0.352828837
$s_0 = 2$	0.073571558	0.069720729	0.069859192	0.01187042	0.225021899
$s_0 = 3$	0.121758013	0.119123591	0.117997992	0.009865938	0.368745534
$s_0 = 4$	0.015051733	0.019018542	0.003734008	0.015599448	0.053403731
$\Pr(s_t)$	0.352828837	0.225021899	0.368745534	0.053403731	

Again, these data indicate when the initial state is known the long-run expected logarithmic return is $\log \frac{1}{\delta} = 0.03454291$ *and the expected gain from initial state information is*

$$\begin{aligned} I(s_t; s_0) &= H(s_t) + H(s_0) - H(s_t, s_0) \\ &= 1.227545054 + 1.227545054 - 2.368419731 \\ &= 0.086670377 \end{aligned}$$

and the benchmark loss from ignoring the information is the difference -0.052127471*.*

The expected gain from the accounting information in isolation remains

$$\begin{aligned} I(s_t; z_a) &= H(s_t) + H(z_a) - H(s_t, z_a) \\ &= 1.227545054 + 0.680976249 - 1.884065075 \\ &= 0.024456227 \end{aligned}$$

or an expected logarithmic return equal to -0.027671244*. However, the accounting information offers no benefit in concert with other information as other information, z, reveals the initial state and accounting information only relates to the initial state.*

Mutual information associated with other information is

$$\begin{aligned} I(s_t; z) &= H(s_t) + H(z) - H(s_t, z) \\ &= 1.227545054 + 1.227545054 - 2.368419731 \\ &= 0.086670377 \end{aligned}$$

or other information generates an expected logarithmic return equal to $\log \frac{1}{\delta} = 0.03454291$ *or* $0.086670377 - 0.052127471 = 0.03454291$*.*

Of course, there is no complementarity of initial state, accounting, and other *information as collectively and individually the information relates purely to the initial state.*

$$\begin{aligned} I(s_t; z_a, z, s_0) &= H(s_t) + H(z_a, z, s_0) - H(s_t, z_a, z, s_0) \\ &= 1.227545054 + 1.227545054 - 2.368419731 \\ &= 0.086670377 \end{aligned}$$

Results for various information profiles are summarized below.

information	$I(s_t; info)$	$E[r \mid info]$
none	0	−0.052127471
s_0	0.086670377	0.034542906
z_a	0.024456227	−0.027671244
z	0.086670377	0.034542906
z_a, z	0.086670377	0.034542906
s_0, z_a	0.086670377	0.034542906
s_0, z	0.086670377	0.034542906
s_0, z_a, z	0.086670377	0.034542906

The next example addresses the question "could coarse, initial state accounting information beneficially complement forward-looking other information?"

Example 3 (complementary other forward-looking information) *Everything remains as in example 1 except the information structure. Accounting and other information inform the initial state through* $\Pr(z_a, z \mid s_0, s_t)$

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.9	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.0	0.9	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.9	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.9	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.9	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.9	0.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.9	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.9	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.1	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.1	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.1	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.1	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.1
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.1
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.1
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.1

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.1	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.1	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.1	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.1	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.1	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.1	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.1	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.1	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.9	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.9	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.0	0.9
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.9	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.9
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.9
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.9	0.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.9

Hence, the joint distribution is $\Pr(z_a, z, s_0, s_t) = \Pr(z_a, z \mid s_0, s_t) \Pr(s_t \mid s_0) \Pr(s_0)$ *where* $\Pr(s_0) = p_{ss}$

$\Pr(z_a, z, s_0, s_t)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.12820278	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.0	0.015443133	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.159438909	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.014461132	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.066214402	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.062748656	0.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.062873272	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.010683378	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.012175801	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.011912359	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.011799799	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.000986594	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.001505173
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.001901854
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.000373401
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.001559945

$\Pr(z_a, z, s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.014244753	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.001715904	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.017715434	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.001606792	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.0	0.007357156	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.006972073	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.0	0.006985919	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.001187042	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.109582211	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.107211232	0.0
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.0	0.106198193
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.008879344	0.0
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.01354656
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.017116688
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.003360607	0.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.014039503

$\Pr(z_a, s_t) = \sum_{z,s_0} \Pr(z_a, z, s_0, s_t)$

$\Pr(z_a, s_t)$	$z_a = 12$	$z_a = 34$	$\Pr(s_t)$
$s_t = 1$	0.208098157	0.14473068	0.352828837
$s_t = 2$	0.092006002	0.133015896	0.225021899
$s_t = 3$	0.234485381	0.134260153	0.368745534
$s_t = 4$	0.027691049	0.025712682	0.053403731
$\Pr(z_a)$	0.562280588	0.437719412	

$\Pr(z, s_t) = \sum_{z_a,s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z, s_t)$	$z = 1$	$z = 2$	$z = 3$	$z = 4$	$\Pr(s_t)$
$s_t = 1$	0.142447533	0.073571558	0.121758013	0.015051733	0.352828837
$s_t = 2$	0.06446456	0.022415206	0.119123591	0.019018542	0.225021899
$s_t = 3$	0.177154343	0.069859192	0.015160406	0.106571593	0.368745534
$s_t = 4$	0.016067924	0.01187042	0.009865938	0.015599448	0.053403731
$\Pr(z)$	0.40013436	0.177716376	0.265907948	0.156241316	

$\Pr(z_a, z, s_t) = \sum_{s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z_a, z, s_t)$	$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$	$\Pr(z_a, z)$
$z_a = 12, z = 1$	0.12820278	0.062748656	0.159438909	0.014461132	0.364851476
$z_a = 12, z = 2$	0.066214402	0.015443133	0.062873272	0.010683378	0.155214186
$z_a = 12, z = 3$	0.012175801	0.011912359	0.011799799	0.000986594	0.036874553
$z_a = 12, z = 4$	0.001505173	0.001901854	0.000373401	0.001559945	0.005340373
$z_a = 34, z = 1$	0.014244753	0.001715904	0.017715434	0.001606792	0.035282884
$z_a = 34, z = 2$	0.007357156	0.006972073	0.006985919	0.001187042	0.02250219
$z_a = 34, z = 3$	0.109582211	0.107211232	0.003360607	0.008879344	0.229033395
$z_a = 34, z = 4$	0.01354656	0.017116688	0.106198193	0.014039503	0.150900943
$\Pr(s_t)$	0.352828837	0.225021899	0.368745534	0.053403731	

Again, these data indicate when the initial state is known the long-run expected logarithmic return is $\log \frac{1}{\delta} = 0.03454291$ *and the expected gain from initial state information is*

$$\begin{aligned} I(s_t; s_0) &= H(s_t) + H(s_0) - H(s_t, s_0) \\ &= 1.227545054 + 1.227545054 - 2.368419731 \\ &= 0.086670377 \end{aligned}$$

and the benchmark loss from ignoring the information is the difference -0.052127471*.*

The expected gain from the accounting information in isolation is lower than the previous examples

$$\begin{aligned} I(s_t; z_a) &= H(s_t) + H(z_a) - H(s_t, z_a) \\ &= 1.227545054 + 0.685369251 - 1.897382779 \\ &= 0.015531526 \end{aligned}$$

or an expected logarithmic return equal to -0.036595945*. However, the accounting information complements other information.*

Mutual information associated with other information is

$$\begin{aligned} I(s_t; z) &= H(s_t) + H(z) - H(s_t, z) \\ &= 1.227545054 + 1.315784008 - 2.403920949 \\ &= 0.139408113 \end{aligned}$$

while mutual information associated with both accounting and other information is

$$\begin{aligned} I(s_t; z_a, z) &= H(s_t) + H(z_a, z) - H(s_t, z_a, z) \\ &= 1.227545054 + 1.632979114 - 2.693502704 \\ &= 0.167021464 \end{aligned}$$

which is greater than the sum of mutual information for z_a *and* z *individually by* 0.012081825*.*

Further, there is rich complementarity of initial state, accounting, and other information

$$\begin{aligned} I(s_t; z_a, z, s_0) &= H(s_t) + H(z_a, z, s_0) - H(s_t, z_a, z, s_0) \\ &= 1.227545054 + 1.959955395 - 2.693502704 \\ &= 0.493997745 \end{aligned}$$

compared with mutual information for the initial state and other information (excluding the accounting information)

$$\begin{aligned} I(s_t; z, s_0) &= H(s_t) + H(z, s_0) - H(s_t, , z, s_0) \\ &= 1.227545054 + 1.658071066 - 2.436235864 \\ &= 0.449380257 \end{aligned}$$

a difference of 0.044617489.

Results for various information profiles are summarized below.

information	$I(s_t; \textit{info})$	$E[r \mid \textit{info}]$
none	0	−0.052127471
s_0	0.086670377	0.034542906
z_a	0.015531526	−0.036595945
z	0.139408113	0.087280641
z_a, z	0.167021464	0.114893992
s_0, z_a	0.086670377	0.034542906
s_0, z	0.449380257	0.397252785
s_0, z_a, z	0.493997745	0.441870274

This suggests that accounting and other information are unlikely to be beneficial to economic agents unless they report the initial state, or more pointedly, are forward-looking indicators of the post-transition state. Next, we turn our attention to forward-looking accounting and other information.

2 Post-transition state information

Suppose accounting information remains coarse but explicitly serves to indicate the post-transition state.

Example 4 (strongly informative accounting) *Everything remains as in example 1 except the information structure. Accounting and other information indicate the post-transition state through* $\Pr(z_a, z \mid s_0, s_t)$

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 12,$ $z = 1$	$z_a = 12,$ $z = 2$	$z_a = 12,$ $z = 3$	$z_a = 12,$ $z = 4$
$s_0 = s_1, s_t = s_1$	0.25	0.25	0.25	0.25
$s_0 = s_1, s_t = s_2$	0.25	0.25	0.25	0.25
$s_0 = s_1, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.25	0.25	0.25	0.25
$s_0 = s_2, s_t = s_2$	0.25	0.25	0.25	0.25
$s_0 = s_2, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.25	0.25	0.25	0.25
$s_0 = s_3, s_t = s_2$	0.25	0.25	0.25	0.25
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_1$	0.25	0.25	0.25	0.25
$s_0 = s_4, s_t = s_2$	0.25	0.25	0.25	0.25
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.0

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.25	0.25	0.25	0.25
$s_0 = s_1, s_t = s_4$	0.25	0.25	0.25	0.25
$s_0 = s_2, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.25	0.25	0.25	0.25
$s_0 = s_2, s_t = s_4$	0.25	0.25	0.25	0.25
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_3$	0.25	0.25	0.25	0.25
$s_0 = s_3, s_t = s_4$	0.25	0.25	0.25	0.25
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_3$	0.25	0.25	0.25	0.25
$s_0 = s_4, s_t = s_4$	0.25	0.25	0.25	0.25

Hence, the joint distribution is $\Pr(z_a, z, s_0, s_t) = \Pr(z_a, z \mid s_0, s_t) \Pr(s_t \mid s_0) \Pr(s_0)$ *where* $\Pr(s_0) = p_{ss}$

$\Pr(z_a, z, s_0, s_t)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.035611883	0.035611883	0.035611883	0.035611883
$s_0 = s_1, s_t = s_2$	0.004289759	0.004289759	0.004289759	0.004289759
$s_0 = s_1, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.018392889	0.018392889	0.018392889	0.018392889
$s_0 = s_2, s_t = s_2$	0.017430182	0.017430182	0.017430182	0.017430182
$s_0 = s_2, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.030439503	0.030439503	0.030439503	0.030439503
$s_0 = s_3, s_t = s_2$	0.029780898	0.029780898	0.029780898	0.029780898
$s_0 = s_3, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_4$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_1$	0.003762933	0.003762933	0.003762933	0.003762933
$s_0 = s_4, s_t = s_2$	0.004754635	0.004754635	0.004754635	0.004754635
$s_0 = s_4, s_t = s_3$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_4$	0.0	0.0	0.0	0.0

$\Pr(z_a, z, s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_1, s_t = s_3$	0.044288586	0.044288586	0.044288586	0.044288586
$s_0 = s_1, s_t = s_4$	0.004016981	0.004016981	0.004016981	0.004016981
$s_0 = s_2, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_2, s_t = s_3$	0.017464798	0.017464798	0.017464798	0.017464798
$s_0 = s_2, s_t = s_4$	0.002967605	0.002967605	0.002967605	0.002967605
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_3, s_t = s_3$	0.029499498	0.029499498	0.029499498	0.029499498
$s_0 = s_3, s_t = s_4$	0.002466484	0.002466484	0.002466484	0.002466484
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.0	0.0
$s_0 = s_4, s_t = s_3$	0.000933502	0.000933502	0.000933502	0.000933502
$s_0 = s_4, s_t = s_4$	0.003899862	0.003899862	0.003899862	0.003899862

$\Pr(z_a, s_t) = \sum_{z,s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z_a, s_t)$	$z_a = 12$	$z_a = 34$
$s_t = 1$	0.352828837	0.0
$s_t = 2$	0.225021899	0.0
$s_t = 3$	0.0	0.368745534
$s_t = 4$	0.0	0.053403731
$\Pr(z_a)$	0.577850736	0.422149264

$\Pr(z, s_t) = \sum_{z_a,s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z, s_t)$	$z = 1$	$z = 2$	$z = 3$	$z = 4$	$\Pr(s_t)$
$s_t = 1$	0.088207209	0.088207209	0.088207209	0.088207209	0.352828837
$s_t = 2$	0.056255475	0.056255475	0.056255475	0.056255475	0.225021899
$s_t = 3$	0.092186383	0.092186383	0.092186383	0.092186383	0.368745534
$s_t = 4$	0.013350933	0.013350933	0.013350933	0.013350933	0.053403731
$\Pr(z)$	0.25	0.25	0.25	0.25	

$\Pr(s_0, s_t) = \sum_{z_a,z} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(s_0, s_t)$	$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$	$\Pr(s_0)$
$s_0 = 1$	0.142447533	0.017159037	0.177154343	0.016067924	0.352828837
$s_0 = 2$	0.073571558	0.069720729	0.069859192	0.01187042	0.225021899
$s_0 = 3$	0.121758013	0.119123591	0.117997992	0.009865938	0.368745534
$s_0 = 4$	0.015051733	0.019018542	0.003734008	0.015599448	0.053403731
$\Pr(s_t)$	0.352828837	0.225021899	0.368745534	0.053403731	

Again, these data indicate when the initial state is known the long-run expected logarithmic return is $\log \frac{1}{\delta} = 0.03454291$ *and the expected gain from initial* *state information is*

$$\begin{aligned} I(s_t; s_0) &= H(s_t) + H(s_0) - H(s_t, s_0) \\ &= 1.227545054 + 1.227545054 - 2.368419731 \\ &= 0.086670377 \end{aligned}$$

and the benchmark loss from ignoring the information is the difference -0.052127471*.*

The expected gain from the accounting information in isolation is quite large

$$\begin{aligned} I(s_t; z_a) &= H(s_t) + H(z_a) - H(s_t, z_a) \\ &= 1.227545054 + 0.680976249 - 1.227545054 \\ &= 0.680976249 \end{aligned}$$

or an expected logarithmic return equal to 0.628848777*.*

However, other information offers no benefit, mutual information associated with other information is

$$\begin{aligned} I(s_t; z) &= H(s_t) + H(z) - H(s_t, z) \\ &= 1.227545054 + 1.386294361 - 2.613839415 \\ &= 0.0 \end{aligned}$$

or other information generates an expected logarithmic return equal to -0.052127471*.*

Of course, there is no complementarity of other information with initial state and accounting information.

$$\begin{aligned} I(s_t; z_a, z, s_0) &= I(s_t; z_a, s_0) \\ I(s_t; z_a, z, s_0) &= H(s_t) + H(z_a, z, s_0) - H(s_t, z_a, z, s_0) \\ &= 1.227545054 + 3.277191784 - 3.754714092 \\ I(s_t; z_a, s_0) &= H(s_t) + H(z_a, s_0) - H(s_t, z_a, s_0) \\ &= 1.227545054 + 1.890897423 - 2.368419731 \\ &= 0.750022746 \end{aligned}$$

Results for various information profiles are summarized below.

information	$I(s_t; \textit{info})$	$E[r \mid \textit{info}]$
none	0	-0.052127471
s_0	0.086670377	0.034542906
z_a	0.680976249	0.628848777
z	0.0	-0.052127471
z_a, z	0.680976249	0.628848777
s_0, z_a	0.750022746	0.697895274
s_0, z	0.086670377	0.034542906
s_0, z_a, z	0.750022746	0.697895274

Next, we explore an example in which post-transition state information (accounting, other, and initial state) are highly complementary. In particular, collectively they benefit the economic agent more than the sum of their individual benefits..

Example 5 (highly complementary accounting and other information) *Everything remains as in example 1 except the information structure. Accounting and other information indicate the post-transition state through* $\Pr(z_a, z \mid s_0, s_t)$

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.3	0.3	0.1	0.1
$s_0 = s_1, s_t = s_2$	0.3	0.3	0.1	0.1
$s_0 = s_1, s_t = s_3$	0.1	0.1	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.1	0.1	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.3	0.3	0.1	0.1
$s_0 = s_2, s_t = s_2$	0.3	0.3	0.1	0.1
$s_0 = s_2, s_t = s_3$	0.1	0.1	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.1	0.1	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.3	0.3	0.1	0.1
$s_0 = s_3, s_t = s_2$	0.3	0.3	0.1	0.1
$s_0 = s_3, s_t = s_3$	0.1	0.1	0.0	0.0
$s_0 = s_3, s_t = s_4$	0.1	0.1	0.0	0.0
$s_0 = s_4, s_t = s_1$	0.3	0.3	0.1	0.1
$s_0 = s_4, s_t = s_2$	0.3	0.3	0.1	0.1
$s_0 = s_4, s_t = s_3$	0.1	0.1	0.0	0.0
$s_0 = s_4, s_t = s_4$	0.1	0.1	0.0	0.0

$\Pr(z_a, z \mid s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.0	0.0	0.1	0.1
$s_0 = s_1, s_t = s_2$	0.0	0.0	0.1	0.1
$s_0 = s_1, s_t = s_3$	0.1	0.1	0.3	0.3
$s_0 = s_1, s_t = s_4$	0.1	0.1	0.3	0.3
$s_0 = s_2, s_t = s_1$	0.0	0.0	0.1	0.1
$s_0 = s_2, s_t = s_2$	0.0	0.0	0.1	0.1
$s_0 = s_2, s_t = s_3$	0.1	0.1	0.3	0.3
$s_0 = s_2, s_t = s_4$	0.1	0.1	0.3	0.3
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.1	0.1
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.1	0.1
$s_0 = s_3, s_t = s_3$	0.1	0.1	0.3	0.3
$s_0 = s_3, s_t = s_4$	0.1	0.1	0.3	0.3
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.1	0.1
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.1	0.1
$s_0 = s_4, s_t = s_3$	0.1	0.1	0.3	0.3
$s_0 = s_4, s_t = s_4$	0.1	0.1	0.3	0.3

Hence, the joint distribution is $\Pr(z_a, z, s_0, s_t) = \Pr(z_a, z \mid s_0, s_t)\Pr(s_t \mid s_0)\Pr(s_0)$ *where* $\Pr(s_0) = p_{ss}$

$\Pr(z_a, z, s_0, s_t)$	$z_a = 12$, $z = 1$	$z_a = 12$, $z = 2$	$z_a = 12$, $z = 3$	$z_a = 12$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.04273426	0.04273426	0.014244753	0.014244753
$s_0 = s_1, s_t = s_2$	0.005147711	0.005147711	0.001715904	0.001715904
$s_0 = s_1, s_t = s_3$	0.017715434	0.017715434	0.0	0.0
$s_0 = s_1, s_t = s_4$	0.001606792	0.001606792	0.0	0.0
$s_0 = s_2, s_t = s_1$	0.022071467	0.022071467	0.007357156	0.007357156
$s_0 = s_2, s_t = s_2$	0.020916219	0.020916219	0.006972073	0.006972073
$s_0 = s_2, s_t = s_3$	0.006985919	0.006985919	0.0	0.0
$s_0 = s_2, s_t = s_4$	0.001187042	0.001187042	0.0	0.0
$s_0 = s_3, s_t = s_1$	0.036527404	0.036527404	0.012175801	0.012175801
$s_0 = s_3, s_t = s_2$	0.035737077	0.035737077	0.011912359	0.011912359
$s_0 = s_3, s_t = s_3$	0.011799799	0.011799799	0.0	0.0
$s_0 = s_3, s_t = s_4$	0.000986594	0.000986594	0.0	0.0
$s_0 = s_4, s_t = s_1$	0.00451552	0.00451552	0.001505173	0.001505173
$s_0 = s_4, s_t = s_2$	0.005705563	0.005705563	0.001901854	0.001901854
$s_0 = s_4, s_t = s_3$	0.000373401	0.000373401	0.0	0.0
$s_0 = s_4, s_t = s_4$	0.001559945	0.001559945	0.0	0.0

$\Pr(z_a, z, s_0, s_t)$	$z_a = 34$, $z = 1$	$z_a = 34$, $z = 2$	$z_a = 34$, $z = 3$	$z_a = 34$, $z = 4$
$s_0 = s_1, s_t = s_1$	0.0	0.0	0.014244753	0.014244753
$s_0 = s_1, s_t = s_2$	0.0	0.0	0.001715904	0.001715904
$s_0 = s_1, s_t = s_3$	0.053146303	0.053146303	0.017715434	0.017715434
$s_0 = s_1, s_t = s_4$	0.004820377	0.004820377	0.001606792	0.001606792
$s_0 = s_2, s_t = s_1$	0.0	0.0	0.007357156	0.007357156
$s_0 = s_2, s_t = s_2$	0.0	0.0	0.006972073	0.006972073
$s_0 = s_2, s_t = s_3$	0.020957757	0.020957757	0.006985919	0.006985919
$s_0 = s_2, s_t = s_4$	0.003561126	0.003561126	0.001187042	0.001187042
$s_0 = s_3, s_t = s_1$	0.0	0.0	0.012175801	0.012175801
$s_0 = s_3, s_t = s_2$	0.0	0.0	0.011912359	0.011912359
$s_0 = s_3, s_t = s_3$	0.035399398	0.035399398	0.011799799	0.011799799
$s_0 = s_3, s_t = s_4$	0.002959781	0.002959781	0.000986594	0.000986594
$s_0 = s_4, s_t = s_1$	0.0	0.0	0.001505173	0.001505173
$s_0 = s_4, s_t = s_2$	0.0	0.0	0.001901854	0.001901854
$s_0 = s_4, s_t = s_3$	0.001120202	0.001120202	0.000373401	0.000373401
$s_0 = s_4, s_t = s_4$	0.004679834	0.004679834	0.001559945	0.001559945

$\Pr(z_a, s_t) = \sum_{z,s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z_a, s_t)$	$z_a = 12$	$z_a = 34$	$\Pr(s_t)$
$s_t = 1$	0.28226307	0.070565767	0.352828837
$s_t = 2$	0.180017519	0.04500438	0.225021899
$s_t = 3$	0.073749107	0.294996427	0.368745534
$s_t = 4$	0.010680746	0.042722984	0.053403731
$\Pr(z_a)$	0.546710441	0.453289559	

$\Pr(z, s_t) = \sum_{z_a,s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z, s_t)$	$z = 1$	$z = 2$	$z = 3$	$z = 4$	$\Pr(s_t)$
$s_t = 1$	0.105848651	0.105848651	0.070565767	0.070565767	0.352828837
$s_t = 2$	0.06750657	0.06750657	0.04500438	0.04500438	0.225021899
$s_t = 3$	0.147498214	0.147498214	0.036874553	0.036874553	0.368745534
$s_t = 4$	0.021361492	0.021361492	0.005340373	0.005340373	0.053403731
$\Pr(z)$	0.342214926	0.342214926	0.157785074	0.157785074	

$\Pr(z_a, z, s_t) = \sum_{s_0} \Pr(z_a, z, s_0, s_t)$ *is*

$\Pr(z_a, z, s_t)$	$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$	$\Pr(z_a, z)$
$z_a = 12, z = 1$	0.105848651	0.06750657	0.036874553	0.005340373	0.215570147
$z_a = 12, z = 2$	0.105848651	0.06750657	0.036874553	0.005340373	0.215570147
$z_a = 12, z = 3$	0.035282884	0.02250219	0.0	0.0	0.057785074
$z_a = 12, z = 4$	0.035282884	0.02250219	0.0	0.0	0.057785074
$z_a = 34, z = 1$	0.0	0.0	0.11062366	0.016021119	0.126644779
$z_a = 34, z = 2$	0.0	0.0	0.11062366	0.016021119	0.126644779
$z_a = 34, z = 3$	0.035282884	0.02250219	0.036874553	0.005340373	0.1
$z_a = 34, z = 4$	0.035282884	0.02250219	0.036874553	0.005340373	0.1
$\Pr(s_t)$	0.352828837	0.225021899	0.368745534	0.053403731	

Again, these data indicate when the initial state is known the long-run expected logarithmic return is $\log \frac{1}{\delta} = 0.03454291$ *and the expected gain from initial state information is*

$$\begin{aligned} I(s_t; s_0) &= H(s_t) + H(s_0) - H(s_t, s_0) \\ &= 1.227545054 + 1.227545054 - 2.368419731 \\ &= 0.086670377 \end{aligned}$$

and the benchmark loss from ignoring the information is the difference -0.052127471.

The expected gain from the accounting information in isolation is substantial

$$\begin{aligned} I(s_t; z_a) &= H(s_t) + H(z_a) - H(s_t, z_a) \\ &= 1.227545054 + 0.68877708 - 1.727947477 \\ &= 0.188374657 \end{aligned}$$

or an expected logarithmic return equal to 0.136247185*.*

Other information alone offers modest benefit, mutual information associated with other information is

$$\begin{aligned} I\left(s_t; z\right) &= H\left(s_t\right) + H\left(z\right) - H\left(s_t, z\right) \\ &= 1.227545054 + 1.316632338 - 2.520837036 \\ &= 0.023340355 \end{aligned}$$

or other information generates an expected logarithmic return equal to -0.028787116*.*

However, there is complementarity among other information, initial state and accounting information, for example, compared with accounting and initial state information.

$$\begin{aligned} I\left(s_t; z_a, z, s_0\right) &= I\left(s_t; z_a, s_0\right) \\ I\left(s_t; z_a, z, s_0\right) &= H\left(s_t\right) + H\left(z_a, z, s_0\right) - H\left(s_t, z_a, z, s_0\right) \\ &= 1.227545054 + 3.193412812 - 4.011837451 \\ &= 0.409120415 \\ I\left(s_t; z_a, s_0\right) &= H\left(s_t\right) + H\left(z_a, s_0\right) - H\left(s_t, z_a, s_0\right) \\ &= 1.227545054 + 1.910063957 - 2.868822154 \\ &= 0.268786857 \end{aligned}$$

Results for various information profiles are summarized below.

information	$I\left(s_t; \textit{info}\right)$	$E\left[r \mid \textit{info}\right]$
none	0	-0.052127471
s_0	0.086670377	0.034542906
z_a	0.188374657	0.136247185
z	0.023340355	-0.028787116
z_a, z	0.331553755	0.279426283
s_0, z_a	0.268786857	0.216659385
s_0, z	0.109206309	0.057078837
s_0, z_a, z	0.409120415	0.356992944