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Depreciation in a model of probabilistic investment

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ABSTRACT

A pervasive theme in both accounting and statistics is aggregation. However, in contrast to statistics, a customary standard for determining the best aggregation rule in accounting is unavailable or, at least, not explicitly defined. Also, most accounting procedures follow a well-specified recursive algorithm of updating a summarized history number (a beginning balance sheet number) by the current period's activities (changes). In this paper, we present a setting in which the best accounting aggregation rule arises naturally, resembles observed depreciation schedules, and proceeds recursively in a manner analogous to the above outlined stock-flow updating process.

Our main results are (1) in every period, the performance of the BLU estimate based on active investments can be replicated by the period's depreciation amount and (2) in every period, the performance of the BLU estimate based on the entire history of investments can be replicated by a recursive procedure that updates the BLU estimate of the previous period with the current period's investment realization. Depreciation successfully satisfies multiple objectives – it serves as a periodic allocation of realized investment amounts and as a statistic for the unknown investment population mean. Depreciation schedules commonly used in practice, straight-line, accelerated and declining balance, are shown to be best in particular settings.

1. INTRODUCTION

Accounting, like statistics, is concerned with aggregation. The trick in statistics is to aggregate sample data so as to create the best guess for a parameter of interest in the underlying population. For example, a sample mean is constructed as an estimate for the unknown population mean. The customary standard for 'best' in statistics is BLU – the best (minimum variance) linear unbiased transformation of observations. In contrast, best aggregation procedures in accounting are harder to define. Also, most accounting procedures follow a well-specified recursive

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algorithm of updating a summarized history number (a beginning balance sheet number) by the current period's activities (changes). The reasons for procedures to unfold over time in this particular fashion are unclear. In this paper, we present a setting in which the best accounting aggregation rule arises naturally, resembles observed depreciation schedules, and proceeds recursively in a manner similar to the stock-flow updating outlined above.

In our model, a firm invests each period, the cash outflow (investment) in each period is observed, and the investment is depreciated according to a predetermined depreciation schedule. The underlying mean of the population from which the investment is drawn is unknown.[1] We assume the decision-maker's goal is to aggregate realized investment amounts to come up with a BLU estimate of the population mean. In our setting, this information can be meaningful to a decision-maker in forecasting next period's investment. Of course, this is not to say that our choice of the stochastic process driven by population mean(s) is the 'correct' process. For example, if investment follows a random walk, then this year's realized investment (cash accounting) is all that is relevant. Our intent is to select a stochastic process that is rich enough to allow for interesting accruals.

We compute two BLU estimates, one that relies only on active (still in use) investments and the other that relies on the entire history of investments. Our main results are (1) in every period, the performance of the 'active' BLU estimate can be replicated by the period's depreciation amount and (2) in every period, the performance of the 'history' BLU estimate can be replicated by a weighted average of the BLU estimate of the previous period and the current period's investment, where the weights are the depreciation rates themselves and the recursive process is initialized by a depreciation amount.

We note three related aspects. First, the best depreciation schedule, one that allows for the replication of the BLU statistic, is unique. Second, the schedule and the recursive weights can be calculated at the outset, since they are independent of investment realizations. Third, depreciation simultaneously satisfies multiple objectives – it serves as a *periodic allocation* of realized investment amounts and as a *statistic* for the unknown investment population mean. According to Thomas (1969, 1974), for allocations to be theoretically justifiable, they must satisfy three criteria. The parts should add to the whole. The method should be unambiguous (unique given the objective function) and defensible (there exists an underlying logic for the objective function that is employed). In our setting, depreciation satisfies these criteria.

Our results suggest that the flexibility (discretion) inherent in the accrual process can be valuable. In particular, the choice of different depreciation schedules enables the firm to tailor the accrual to the circumstance. In the constant mean case, the best depreciation schedule is straight-line. When the mean varies stochastically, it is accelerated. Further, best depreciation can be approximated using the declining balance method: in each period the beginning book value of the investments plus the period's new investment are depreciated at a constant, group rate. We find it reassuring that not only is there a demand for depreciation in our setting, but that it resembles depreciation methods prevalent in practice and commonly taught in classrooms (e.g., straight-line, accelerated, declining balance).

The intuition for when straight-line and accelerated depreciation are optimal can be obtained by studying the effect of non-stationarity in the population mean on the OLS (ordinary least squares) problem. When the population mean is stationary, all past observations (investment realizations) are equally reliable readings of the underlying mean. The optimal estimate is found by weighting the observations equally. The equal-weighting scheme is achieved by using straight-line depreciation. On the other hand, when the mean is non-stationary, the most recent observation is a more reliable estimate of the current population mean than previous observations. As in OLS, current and past observations are used. However, to account for the drift in the population mean, the more recent observations are weighted more heavily in order to form the best statistic. This weighting scheme (also referred to as Kalman filter weights) is achieved by using accelerated depreciation.[2]

There are two sources of noise (error) in our setting. One, the investment realization can differ from the population mean due to noise. Two, the population mean itself can vary from period to period because of stochastic drift. For the benchmark case, where these error terms have the same variance, the best depreciation schedule has an aesthetic property but not one commonly encountered by accountants. The best depreciation schedule is comprised of Fibonacci numbers. A Fibonacci sequence is one that starts with 0 (F_0) and 1 (F_1) and each successive number is formed by the sum of the previous two numbers. The fact that Fibonacci numbers show up in a stylized example does not seem too surprising given that they regularly appear in a variety of unrelated problems.[3] The point, of course, is not that Fibonacci numbers can show up in the best depreciation schedule. Rather the point is that the best depreciation schedule depends in a subtle fashion on the underlying cash outflow process.

The fact that accruals can be valuable in conveying information has been stressed in the literature. Demski and Sappington (1990) treat income measurement as a process by which useful information is conveyed, using the language of income measurement or asset valuation. In their setting it is always possible to construct an accounting treatment that fully discloses the underlying (unrecorded) cash flow stream. Brief and Owen (1970, 1973), focus on cost assignment to obtain a number called 'period income' which can be translated into an estimate of a financial parameter associated with an asset such as its internal rate of return. In a single-period inference problem, Arya *et al.* (2000b, 2000c) derive the 'best guess' of the transactions that underlie a given set of financial statements. This derivation is simplified by the accountant's use of double entry. Christensen and Demski (1995) study a control problem in which audited accruals (in particular, audited depreciation numbers) discipline other more timely sources of information and also carry information about the environment and the agent's behaviour.

The remainder of the paper is organized as follows. Section 2 specifies the cash outflow process. Section 3 derives the BLU statistic and interprets it as depreciation. Section 4 concludes the paper.

2. CASH OUTFLOW PROCESS

A firm purchases an asset in every period t, $t = 1, 2$, etc. Each asset has a useful life of T periods. The purchase price (investment or cash outflow) of the asset in period t is denoted I_t, where I_t is drawn from an underlying population with mean $\bar{I}_t$. This process is formally represented as follows:

$$I_t = \bar{I}_t + e_t \tag{1}$$

where e_t is a mean 0 and variance σ_e^2 noise term.

Additionally, the investment population mean, $\bar{I}_t$, is unknown, and is itself generated by a stochastic process, with (deterministic) growth term g, $g > 0$, as follows:

$$\bar{I}_t = g\bar{I}_{t-1} + \varepsilon_t, \quad t > 1 \tag{2}$$

where ε_t is a mean 0 and variance $\sigma_\varepsilon^2 = \sigma_e^2/v^2$ noise term. The error terms, e_i, e_j, ε_i, ε_j, $i, j = 1, 2, \ldots, i \neq j$, are assumed to be mutually independent. The parameters governing the stochastic processes in (1) and (2), g, σ_e^2 and σ_ε^2, are known. Since the mean can itself vary stochastically over time, the process is termed non-stationary. A degenerate case with $\sigma_\varepsilon^2 = 0$ is termed stationary, since it implies a non-stochastic mean.

The conventional accounting treatment of these cash outflows is to record them as an asset and depreciate them over T periods, their useful life.[4] Denote the depreciation (rate) schedule by $(d_1^T, d_2^T, \ldots, d_T^T)$. Each investment, say investment I_t, is depreciated an amount $d_1^T I_t$ in its first year of existence, $d_2^T I_t$ in its second year and so on. For d_k^T, $k = 1, 2, \ldots, T$, to be a valid depreciation schedule, we require it to be non-negative ($d_k^T \geqslant 0$) and tidy ($\sum_{k=1}^T d_k^T = 1$).

The depreciation in period t is denoted D_t. We focus on the properties of the depreciation number after the firm has reached steady state, i.e., $t \geqslant T$.[5] In steady state, D_t is calculated as follows:

$$D_t = d_1^T I_t + d_2^T I_{t-1} + \cdots + d_T^T I_{t-T-1}, \quad t \geqslant T$$

We refer to investments $I_t, I_{t-1}, \ldots, I_{t-T+1}$, used in the depreciation calculation as active investments in period t. The entire history of investments, $I_t, I_{t-1}, \ldots, I_1$, includes both active and inactive investments.

The following notation will prove convenient in presenting our results. For all positive integers k, define a_k as

$$\left(\frac{2v}{\sqrt{1+4v^2}-1}\right)^{-2k} + \left(\frac{2v}{\sqrt{1+4v^2}-1}\right)^{2(k-1)}$$

For crispness, we initially present our results for the $g = 1$ case, i.e., the zero-growth steady-state problem. Later, we show our results hold for any choice of growth.

3. RESULTS

Depreciation as a BLU statistic

In this section, we derive the BLU estimate in each period assuming only active investments, i.e., the most recent T observations, are used. We then check to see if the performance of this BLU statistic can be replicated by the period's depreciation number for some suitably chosen depreciation schedule.

The trick is to write the system of equations governing cash outflows in a form that allows for OLS to be used to determine the BLU estimate. For a given T, say $T = 4$, the system of linear equations in (1) and (2) in period t ($t \geqslant 4$) are listed below. Note two things. One, since each investment's useful life is four periods, the only active investments in period t are I_t, I_{t-1}, I_{t-2} and I_{t-3}. Two, we have multiplied the equations in (2) by v. The advantage of doing so will soon be apparent.

$$
\begin{aligned}
I_{t-3} &= \bar{I}_{t-3} + e_{t-3} \\
0 &= -\mathrm{v}\bar{I}_{t-3} + \mathrm{v}\bar{I}_{t-2} - \mathrm{v}\varepsilon_{t-2} \\
I_{t-2} &= \bar{I}_{t-2} + e_{t-2} \\
0 &= -\mathrm{v}\bar{I}_{t-2} + \mathrm{v}\bar{I}_{t-1} - \mathrm{v}\varepsilon_{t-1} \\
I_{t-1} &= \bar{I}_{t-1} + e_{t-1} \\
0 &= -\mathrm{v}\bar{I}_{t-1} + \mathrm{v}\bar{I}_{t} - \mathrm{v}\varepsilon_{t} \\
I_{t} &= \bar{I}_{t} + e_{t}
\end{aligned}
$$

In matrix form, $I = H\bar{I} + \eta$, where

$$
I = \begin{bmatrix} I_{t-3} \\ 0 \\ I_{t-2} \\ 0 \\ I_{t-1} \\ 0 \\ I_t \end{bmatrix}, \quad
H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\mathrm{v} & \mathrm{v} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\mathrm{v} & \mathrm{v} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\mathrm{v} & \mathrm{v} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad
\eta = \begin{bmatrix} e_{t-3} \\ -\mathrm{v}\varepsilon_{t-2} \\ e_{t-2} \\ -\mathrm{v}\varepsilon_{t-1} \\ e_{t-1} \\ -\mathrm{v}\varepsilon_t \\ e_t \end{bmatrix} \quad \text{and}
$$

$$
\bar{I} = \begin{bmatrix} \bar{I}_{t-3} \\ \bar{I}_{t-2} \\ \bar{I}_{t-1} \\ \bar{I}_t \end{bmatrix}
$$

Owing to the multiplication of (2) by v, the error vector η has mean 0 and its covariance matrix is σ_e^2 times the identity matrix. Hence, OLS yields the BLU estimate for the vector $\bar{I}$. The OLS estimate for $\bar{I}$ is $(H'H)^{-1}H'I$, and the covariance matrix associated with this estimate is $\sigma_e^2(H'H)^{-1}$. We are interested in the BLU estimate for $\bar{I}_t$ denoted $\hat{\bar{I}}_t$. This estimate is the last element in the $(H'H)^{-1}H'I$ vector, and its variance is the lower diagonal element in the $\sigma_e^2(H'H)^{-1}$ matrix. But what does the $\hat{\bar{I}}_t$ estimate have to do with depreciation?

To answer the above question, let us calculate the BLU estimate for two particular settings: a non-stationary setting with $\sigma_e^2 = \sigma_\varepsilon^2 = 1$ and a stationary setting with $\sigma_e^2 = 1$ and $\sigma_\varepsilon^2 = 0$. The BLU estimate for $\bar{I}_t$ in the non-stationary case is obtained by plugging $v = 1$ in $\hat{\bar{I}}_t$. The corresponding estimate in the stationary case is obtained by taking the limit of $\hat{\bar{I}}_t$ as $v \to \infty$. These solutions are listed below.

- The $\sigma_e^2 = 1$ and $\sigma_\varepsilon^2 = 1$ case: $\hat{\bar{I}}_t = \frac{13}{21}I_t + \frac{5}{21}I_{t-1} + \frac{2}{21}I_{t-2} + \frac{1}{21}I_{t-3}$ and the variance of $\hat{\bar{I}}_t$ is $\frac{13}{21}$.
- The $\sigma_e^2 = 1$ and $\sigma_\varepsilon^2 = 0$ case: $\hat{\bar{I}}_t = \frac{1}{4}I_t + \frac{1}{4}I_{t-1} + \frac{1}{4}I_{t-2} + \frac{1}{4}I_{t-3}$ and the variance of $\hat{\bar{I}}_t$ is $\frac{1}{4}$.

In either case, the BLU weights are non-negative and sum to one. Hence, these are valid depreciation rates. If depreciation is computed using these rates, not only will each investment be fully depreciated over its useful life, it will also communicate the 'active' BLU estimate to the reader in every period.

Furthermore, the depreciation methods are distinct in the two cases. In the stationary case, depreciation is straight-line: $\frac{1}{4}$ of the investment is depreciated each period. In the non-stationary case, depreciation is accelerated: $\frac{13}{21}$ of the investment is depreciated in the first period, $\frac{5}{21}$ in the second, and so on. In the stationary case, all observations are equally reliable and, hence, the BLU estimate is simply the equally weighted sample mean. In the non-stationary case, the more recent the observation, the fewer the number of ε-shocks separating it from the current population mean. For example, the current observation (I_t) differs from the current mean ($\bar{I}_t$) only due to an e-shock (e_t). On the other hand, the previous observation (I_{t-1}) differs from the current mean due to both an e-shock (e_{t-1}) and an ε-shock (ε_t), the latter being associated with the drift in the mean and so on. The use of an accelerated depreciation schedule ensures that the recent, more reliable, observations are weighted more.

The above argument suggests that as v changes, the amount of acceleration in the optimal depreciation schedule should also change in order to replicate the BLU statistic. As our proposition states, this conjecture is correct. (All proofs are presented in the Appendix.)

Proposition 1.

(a) The BLU estimator in period t obtained using the T most recent investment realizations is the depreciation amount D_t^* computed using the rate schedule d_k^{T*}, $d_k^{T*} = a_{T-k+1}/[\nu^2(a_{T+1} - a_T)]$.
(b) The best depreciation schedule is accelerated ($d_1^{T*} > d_2^{T*} > \cdots > d_T^{T*}$) in the non-stationary case and straight-line ($d_1^{T*} = d_2^{T*} = \cdots = d_T^{T*}$) in the stationary case.
(c) The variance associated with using the estimate D_t^* is $d_1^{T*}\sigma_\epsilon^2$.

We have already discussed the intuition for parts (a) and (b) of the proposition. The intuition behind part (c) is as follows. Recall, $\hat{I}_t$ is the last element in the vector $(H'H)^{-1}H'I$. The vector $H'I$ is simply the vector of investment amounts $(I_{t-T+1}, \ldots, I_t)$. Hence, $\hat{I}_t$ is a linear combination of these investment amounts, in which I_t is multiplied by the lower diagonal element in $(H'H)^{-1}$. But, recall again, this is exactly where the variance of the estimate resides (times σ_ϵ^2). A quick glance at our $T = 4$ examples confirms these findings. Since $\sigma_\epsilon^2 = 1$ in our examples, the coefficient on I_t ($\frac{13}{21}$ and $\frac{1}{4}$, respectively) are also the variances of the estimates.

The accelerated depreciation schedule that arises when $\sigma_\epsilon^2 = \sigma_\varepsilon^2$ deserves comment. In this case, the numerators and denominators in the fractions that comprise the best depreciation schedule are Fibonacci numbers for any choice of T. In our $T = 4$ example, the best depreciation schedule is $(\frac{13}{21}, \frac{5}{21}, \frac{2}{21}, \frac{1}{21})$, and this can be written as $(\mathrm{F}_7/\mathrm{F}_8, \mathrm{F}_5/\mathrm{F}_8, \mathrm{F}_3/\mathrm{F}_8, \mathrm{F}_1/\mathrm{F}_8)$, where $(\mathrm{F}_0, \mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4, \ldots) = (0, 1, 1, 2, 3, \ldots)$ is the Fibonacci sequence $\mathrm{F}_t = \mathrm{F}_{t-1} + \mathrm{F}_{t-2}$. Corollary 1 states this result formally.

Corollary 1. If the investment population is non-stationary with $\sigma_\epsilon^2 = \sigma_\varepsilon^2$, the best depreciation schedule is Fibonacci, i.e., $d_k^{T*} = \mathrm{F}_{2(T-k)+1}/\mathrm{F}_{2T}$.

Recursive updating of the BLU statistic[6]

In the analysis so far we used only active investments in creating the BLU statistic. This obviously ignores realizations which occurred more than T periods ago. In this section, we allow for the statistic to be conditioned on the entire history of investment realizations. Two questions arise. One, how should active investments be combined with past investment realizations to construct an efficient statistic? Two, what is the cost of ignoring history?

We begin by addressing the first question. Denote the BLU estimator in period t, $t \geqslant T$, obtained using the entire history of t investment realizations by BLU_t. Assume $T = 3$ and $\sigma_\epsilon^2 = \sigma_\varepsilon^2 = 1$. Using the Fibonacci characterization in Corollary 1, we know the best depreciation schedule $(d_1^{3*}, d_2^{3*}, d_3^{3*})$ is $(\frac{5}{8}, \frac{2}{8}, \frac{1}{8})$. When $t = 3$, from Proposition 1, BLU_3 is simply D_3^*, since all investments are active. In the fourth period after I_4 is observed, the BLU_4 weights are

$(d_1^{4*}, d_2^{4*}, d_3^{4*}, d_4^{4*}) = (\frac{13}{21}, \frac{5}{21}, \frac{2}{21}, \frac{1}{21})$. The question is: can we construct BLU_4 by simply updating BLU_3 using I_4 alone or do we need to access I_1, I_2 and I_3?

Consider a weighting scheme that places weight $d_1^{4*} = \frac{13}{21}$ on I_4 and the remaining $(1 - d_1^{4*}) = \frac{8}{21}$ weight on BLU_3. By construction, the weight on I_4 is the same as under BLU_4. Under our updated weighting scheme the weights on I_3, I_2 and I_1 are $(1 - d_1^{4*})d_1^{3*} = (\frac{8}{21})(\frac{5}{8}) = \frac{5}{21}$, $(1 - d_1^{4*})d_2^{3*} = (\frac{8}{21})(\frac{2}{8}) = \frac{2}{21}$, and $(1 - d_1^{4*})d_3^{3*} = (\frac{8}{21})(\frac{1}{8}) = \frac{1}{21}$, respectively. These are exactly the same weights as under BLU_4. Further, these weights have been generated using only the best estimate from the previous period, the current observation, and d_1^{4*}. The next proposition states that this recursive process works for any t, T, σ_c^2 and σ_ε^2.

> **Proposition 2.** The BLU estimator in period t, $t > T$, obtained using the entire history of investment realizations is BLU_t, where BLU_t is computed recursively as follows:
>
> $$BLU_t = (1 - d_1^{t*})BLU_{t-1} + d_1^{t*}I_t$$
>
> where $BLU_T = D_T^*$.

The relative weight on the current realization versus history is consistent with intuition. The longer the history, the less the impact of the current observation. For example, if $T = 3$ and $\sigma_c^2 = \sigma_\varepsilon^2$, the weight on I_4 in period 4 is $\frac{13}{21}$ which is less than $\frac{5}{8}$, the weight on I_3 in period 3.

A simple depreciation technique, the declining balance method, can be used to approximate BLU_t. Under the declining balance method, depreciation is calculated by multiplying the book value of past investments and the current investment by a constant (independent of t and investment realizations) rate. Suppose $\sigma_c^2 = \sigma_\varepsilon^2$, and a declining balance rate of $r = 2/[\sqrt{5} + 1] \approx 0.618$ is used to calculate depreciation. The BLU estimate and depreciation in period 4 are:

$$\begin{aligned}
BLU_4 &= (\tfrac{1}{21})I_1 + (\tfrac{2}{21})I_2 + (\tfrac{5}{21})I_3 + (\tfrac{13}{21})I_4 \\
&= 0.048I_1 + 0.095I_2 + 0.238I_3 + 0.619I_4 \\
D_4 &= (1 - r)^3 rI_1 + (1 - r)^2 rI_2 + (1 - r)rI_3 + rI_4 \\
&= 0.034I_1 + 0.090I_2 + 0.236I_3 + 0.618I_4
\end{aligned}$$

The declining balance weights are approximately the same as the BLU weights. In fact, as t gets larger, the approximation improves and, in the limit, is identical. A quick way to see this limiting result is as follows. When $\sigma_c^2 = \sigma_\varepsilon^2$, the BLU weight on I_t as $t \to \infty$ approaches the inverse of the golden ratio, which is simply r.[7] By definition, the declining balance method too imposes the weight r on the most recent cash flow.[8]

This limiting result holds generally. And, in each case, the best declining balance depreciation rate, denoted r^*, is determined by taking the limit of d_1^{t*} as $t \to \infty$. This yields $r^* = 2/[\sqrt{1 + 4v^2} + 1]$.

Corollary 2. For large t, declining balance depreciation with rate r^* can be used to approximate BLU_t. The error in approximation approaches zero as t approaches infinity.

Propositions 1 and 2 have shown that the best estimator in period t is D_t^*, if only the T active investments are used, and BLU_t, if the entire history of investment realizations is used. From Proposition 1(c), the efficiency loss (the percentage increase in the variance of the estimator) associated with using T rather than t investment realizations is $(d_1^{T*}/d_1^{t*}) - 1$. (If all t realizations are used, the BLU estimate and its variance can be obtained from Proposition 1 by replacing T by t.) An upper bound on this efficiency loss corresponds to its limit when t approaches infinity, since this implies the maximum number of observations are ignored. From Corollary 2, this bound is $(d_1^{T*}/r^*) - 1$.

Another bound on the efficiency loss is obtained by noting that the loss is at a maximum in the stationary case. Ignoring early observations is most costly in the stationary case since these observations, when available, are weighted just as much as recent observations (the BLU weights are straight-line). In contrast, in the non-stationary case, the BLU weights decline over time (the BLU weights are accelerated) and, hence, ignoring past observations is less costly. These results are summarized in the next corollary.

Corollary 3. The efficiency loss is increasing in v and is bounded by

$$\min\left\{\frac{d_1^{T*}}{r^*} - 1, \frac{t}{T} - 1\right\}$$

Incorporating growth

In this section we derive results assuming a non-trivial deterministic growth in the population mean. In this case depreciation continues to be a sufficient statistic for the population mean. However, unlike before, depreciation is no longer the best estimate of the mean. The best estimate is obtained by a scalar transformation (independent of the investment realizations) of the depreciation number. The scalar transformation of the accrual (a sufficient statistic) serves the same purpose as $1/T$ serves for the sum of T investment realizations (also a sufficient statistic) in the zero-growth stationary case. The analogue of the two propositions when the growth term is included is as follows. (Closed-form expressions for the scalar multiple and for the depreciation rates, which now depend on the growth rate, are provided in the proof of Corollary 4.)

Corollary 4. For any growth rate g:

(i) The BLU estimator in period t obtained using the T most recent investment realizations is a constant multiple (m_T) of the depreciation amount D_t^*.

(ii) The BLU estimator in period t, $t > T$, obtained using the entire history of investment realizations is BLU_t, where BLU_t is computed recursively as follows:

$$\mathrm{BLU}_t = \frac{m_t}{m_{t-1}}(1 - d_1^{t*})\mathrm{BLU}_{t-1} + m_t d_1^{t*} I_t$$

where $\mathrm{BLU}_T = m_t D_T^*$.

The delicate dependence of the best depreciation schedule on the stochastic process is further highlighted by revisiting the stationary setting but with $g \neq 1$. In this case, the best depreciation schedule is not straight-line. To see this, note that the system of linear equations in period t in the stationary case is of the form $I_{t-k+1} = \bar{I}_t/g^{k-1} + e_{t-k-1}$, where k runs from 1 to T, if only active items are used, and from 1 to t, if the entire history is used. These equations can be rewritten as $g^{k-1}I_{t-k-1} = \bar{I}_t + g^{k-1}e_{t-k+1}$. This rewritten form stresses that if the observations are scaled so as to have the same mean $\bar{I}_t$, the scaled variables have unequal reliability. The BLU statistic is straightforward to construct – the scaled variables are weighted in the inverse proportion of their variances. This implies the ratio of the BLU weight on I_t versus I_{t-1} is g, I_t versus I_{t-2} is g^2, and so on.

If a depreciation schedule is to replicate the performance of the BLU statistic, it must maintain the same proportion of weights. The best depreciation schedule identified in Corollary 4 accomplishes this in the stationary case by setting $d_2^{T*} = d_1^{T*}/g$, $d_3^{T*} = d_1^{T*}/g^2$, and so on. (d_1^{T*} is calculated to ensure depreciation is tidy.) For example, if $T = 4$ and $g = 2$, the best depreciation schedule in the stationary case is $\frac{8}{15}, \frac{4}{15}, \frac{2}{15}$ and $\frac{1}{15}$. Note that when $g > 1$, the best depreciation schedule is accelerated even in the stationary case. Finally, the BLU statistic for the example is obtained using a simple rescaling of depreciation by $m_4 = \frac{24}{17}$.

4. CONCLUSION

In this paper, we present a setting in which the best accounting aggregation rule arises naturally, resembles observed depreciation schedules, and proceeds recursively in a simple manner. In particular, we show that the performance of the BLU estimate based on active investments in any period can be replicated by the period's depreciation amount. Moreover, in every period, the performance of the BLU estimate based on the entire history of investments can be replicated by a recursive procedure that updates the BLU estimate of the previous period with the current period's investment realization. Depreciation successfully satisfies multiple objectives – it serves as a periodic allocation of realized investment amounts and as a statistic for the unknown investment population mean. Depreciation schedules commonly used in practice, straight-line, accelerated and declining balance, are shown to be best in particular settings.

Statistical results have been important in a variety of accounting-related research endeavours. Empirical accounting research routinely makes use of statistical methods. The equivalence between informativeness, conditional controllability and sufficient statistics forms the underpinnings of many of the agency results in accounting.[9] In this paper we emphasize the fact that accrual numbers can themselves be treated as statistical estimates of unknown population parameters. We further emphasize that the statistical role of depreciation need not compromise its traditional use as an allocation device.

When depreciation is discussed, a common criterion is to choose a depreciation method that best matches costs with revenues. Conceptually the matching theory is powerful. However, operationalizing it can sometimes be problematic. In our setting, matching cannot be operationalized, since we consider only one cash stream, the cash outflow; there is no inflow stream with which to match.[10] One way to view the paper is that it suggests there may be reasons to choose a particular depreciation method even when matching cannot be operationalized. An extension of this work would be to explicitly recognize both a cash inflow and a cash outflow stream and study the tension that arises (if any) between the objectives of matching and providing the best estimate of a chosen population parameter. The explicit modelling of cash inflows may also permit a Feltham–Ohlson-style analysis in which the role of individual accruals may be further explored.

In this paper, a demand for the BLU statistic is assumed. An extension would be to identify decision or control settings in which a demand for the BLU statistic arises endogenously. Further, even if a larger setting can be defined in which a demand for such estimation arises, it raises the following question: when, if ever, will a decision-maker strictly prefer having access to the BLU statistic rather than make use of the entire set of investment realizations. Agency models have studied settings in which the principal's limited ability to commit leads to a demand for coarse information (see, for example, Arya *et al.*, 2000a; Cremer, 1995; Sappington, 1986). Whether or not there are natural agency settings in which the optimal aggregation rule takes the form of recognizable depreciable schedules is an open question.

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APPENDIX

Proof of Proposition 1

Proof of (a). Consider the system of linear equations in period t, when the BLU estimate is constructed using the most recent T observations.

$$
\begin{aligned}
I_{t-T+1} &= \bar{I}_{t-T+1} + e_{t-T+1} \\
0 &= -\nu\bar{I}_{t-T+1} + \nu\bar{I}_{t-T+2} - \nu\varepsilon_{t-T+2} \\
&\;\;\vdots \\
I_{t-2} &= \bar{I}_{t-2} + e_{t-2} \\
0 &= -\nu\bar{I}_{t-2} + \nu\bar{I}_{t-1} - \nu\varepsilon_{t-1} \\
I_{t-1} &= \bar{I}_{t-1} + e_{t-1} \\
0 &= -\nu\bar{I}_{t-1} + \nu\bar{I}_{t} - \nu\varepsilon_{t} \\
I_{t} &= \bar{I}_{t} + e_{t}
\end{aligned}
$$

The reason for multiplying the ε-error equations with ν is to ensure that the error term in all equations has mean zero and variance σ_e^2. In matrix form the above equations can be written as $I = H\bar{I} + \eta$, where

$$
I = \begin{bmatrix} I_{t-T+1} \\ 0 \\ \vdots \\ I_{t-2} \\ 0 \\ I_{t-1} \\ 0 \\ I_t \end{bmatrix}, \quad
H = \begin{bmatrix}
1 & 0 & \dots & 0 & 0 & 0 \\
-\nu & \nu & \dots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 \\
0 & 0 & \dots & -\nu & \nu & 0 \\
0 & 0 & \dots & 0 & 1 & 0 \\
0 & 0 & \dots & 0 & -\nu & \nu \\
0 & 0 & \dots & 0 & 0 & 1
\end{bmatrix},
$$

$$
\eta = \begin{bmatrix} e_{t-T+1} \\ -\nu\varepsilon_{t-T+2} \\ \vdots \\ e_{t-2} \\ -\nu\varepsilon_{t-1} \\ e_{t-1} \\ -\nu\varepsilon_t \\ e_t \end{bmatrix} \quad \text{and} \quad
\bar{I} = \begin{bmatrix} \bar{I}_{t-T+1} \\ \bar{I}_{t-T+2} \\ \vdots \\ \bar{I}_{t-2} \\ \bar{I}_{t-1} \\ \bar{I}_t \end{bmatrix}
$$

The $2T-1$ length error vector η has mean zero and its variance–covariance matrix is σ_e^2 times the identity matrix. The BLU estimator for $\bar{I}$ is

$\hat{\bar{I}} = (H'H)^{-1}H'I$. The last element in the vector $\hat{\bar{I}}$ is the BLU estimator for $\bar{I}_t$. Since the operation $H'I$ yields a vector of the most recent T investment realizations, the BLU weights on the T investment realizations reside in the last row of $(H'H)^{-1}$. We next prove this row is

$$[d_T^{T*} \quad d_{T-1}^{T*} \quad \cdots \quad d_2^{T*} \quad d_1^{T*}]$$

Given H, the matrix $H'H = [h_{ij}]$, $i, j = 1, \ldots, T$, is tridiagonal and is as follows:

$$\begin{aligned} h_{ii} &= 1 + v^2, && \text{if } i = 1 \text{ or } T \\ &= 1 + 2v^2, && \text{otherwise} \\ h_{ij} &= -v^2, && \text{if } j = i + 1, i = 1, \ldots, T-1 \\ &= -v^2, && \text{if } j = i - 1, i = 2, \ldots, T \\ &= 0, && \text{otherwise} \end{aligned}$$

By definition, a necessary condition for

$$[d_T^{T*} \quad d_{T-1}^{T*} \quad \cdots \quad d_2^{T*} \quad d_1^{T*}]$$

to be the last row in $(H'H)^{-1}$ is that

$$[d_T^{T*} \quad d_{T-1}^{T*} \quad \cdots \quad d_2^{T*} \quad d_1^{T*}](H'H)$$

yields $[0 \ 0 \ \cdots \ 0 \ 1]$, the last row in the identity matrix. This condition is also sufficient because $H'H$ has independent rows and columns and, thus, there is only a unique vector (set of weights) such that the dot product of the vector and $H'H$ is $[0 \ 0 \ \cdots \ 0 \ 1]$. (The matrix $H'H$ has independent rows and columns since, clearly, H has independent columns.)

Plugging the closed-form expressions for d_k^{T*} provided in the text it is straightforward to verify that

$$[d_T^{T*} \quad d_{T-1}^{T*} \quad \cdots \quad d_2^{T*} \quad d_1^{T*}](H'H) = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

i.e., the following equations hold:[11]

$$d_T^{T*}(1 + v^2) + d_{T-1}^{T*}(-v^2) = 0$$

$$d_{T-k}^{T*}(-v^2) + d_{T-k-1}^{T*}(1 + 2v^2) + d_{T-k-2}^{T*}(-v^2) = 0, \quad k = 0, 1, \ldots, T-3$$

$$d_2^{T*}(-v^2) + d_1^{T*}(1 + v^2) = 1$$

The final step is to interpret d_k^{T*} as a depreciation rate, i.e., $d_k^{T*} > 0$ for $k = 1, 2, \ldots, T$ and $d_1^{T*} + d_2^{T*} + \cdots + d_{T-1}^{T*} + d_T^{T*} = 1$. These results are a direct consequence of the following:

$$a_k > 0, k = 1, \ldots, T, \quad a_{T+1} > a_T \quad \text{and} \quad a_1 + \cdots + a_T = v^2(a_{T+1} - a_T)$$

Proof of (b). The stationary case corresponds to $\sigma_\eta^2 = 0$. Hence, the best depreciation rate in this case can be obtained by taking the limit of d_k^{T*} as v approaches infinity. This yields $d_k^{T*} = 1/T$. This schedule is straight-line.

In the non-stationary case (when $\sigma_\eta^2 > 0$), the best depreciation schedule d_k^{T*} is accelerated since $a_{T-k+1} - a_{T-(k+1)+1} < 0$ for $k = 1, 2, \ldots, T$. (Recall, a_{T-k+1} is the numerator in d_k^{T*}.)

Proof of (c). The BLU estimator for $\bar{I}$ is $\hat{\bar{I}} = (H'H)^{-1}H'I$. The variance of the estimate $\hat{\bar{I}}$ is $\sigma_\epsilon^2(H'H)^{-1}H'((H'H)^{-1}H')' = \sigma_\epsilon^2(H'H)^{-1}$. The last element in the vector $\hat{\bar{I}}$ is the BLU estimator for $\bar{I}_t$. The variance of this estimate is, hence, the lower right-hand element of $(H'H)^{-1}$ times σ_ϵ^2. The proof is complete since we have already shown that the lower right-hand element of $(H'H)^{-1}$ is d_1^{T*}. □

Proof of Corollary 1

The proof follows immediately by plugging $v = 1$ in d_k^{T*}. □

Proof of Proposition 2

By construction, the weight on I_t in BLU_t is optimal (best). The proof is complete if we can show that $(1 - d_1^{t*})\text{BLU}_{t-1}$ puts optimal weights on all other investment realizations. This is indeed the case. This follows from the fact that $(1 - d_1^{t+1*})d_k^{t*}$ equals d_{k+1}^{t-1*} for $t = T, T+1$, etc., and $k = 1, 2, \ldots, t$. □

Proof of Corollary 2

From the proof of Proposition 2, we know BLU_t puts a weight of d_k^{t*} on I_{t-k+1}. As t approaches infinity, (i) the limit of d_1^{t*} approaches $r^* = 2/[\sqrt{1 + 4v^2} + 1]$ and (ii) the limit of d_k^{t*} equals $d_1^{t*}(1 - d_1^{t*})^{k-1}$, for $k = 2, 3$, etc.

This is a declining balance schedule. The beginning book value of the investment is depreciated each period at the rate r^*. To elaborate, say the depreciable asset cost \$1. In the first period it is depreciated r^*. In the second period the book value of the asset is $(1 - r^*)$. This is depreciated at a rate r^*. In the third period the book value of the asset is $(1 - r^*) - r^*(1 - r^*) = (1 - r^*)^2$. Again, this is depreciated at the rate r^* and so on. □

Proof of Corollary 3

Given Proposition 1(c), the efficiency loss is $d_1^{T*}/d_1^{t*} - 1$. Plugging the closed-form expressions for depreciation from Proposition 1(a), and using some tedious algebra verifies that the efficiency loss is monotonic in v.

From the above, an upper bound on the efficiency loss is obtained when v approaches infinity. This upper bound corresponds to $t/T - 1$, the efficiency loss in the stationary case.

Another upper bound on the efficiency loss corresponds to the limit of $d_1^{T*}/d_1^{t*} - 1$ when t approaches infinity, since this implies the maximum number of observations are ignored. From Corollary 2, this bound is $d_1^{T*}/r^* - 1$. Hence, the $\min\{d_1^{T*}/r^* - 1, t/T - 1\}$ is an upper bound on the efficiency loss. □

Proof of Corollary 4

In this proof we present closed-form expressions for d_k^{T*} and m_T. The reader can then complete the proof by following precisely the same steps as in the proofs of Proposition 1(a) and Proposition 2.

Define

$$c = g + 1/g + 1/(gv^2), \quad \lambda = 0.5(c + \sqrt{c^2 - 4})$$

and

$$b_k = (g\lambda - 1)\lambda^k + \lambda(\lambda - g)\lambda^{-k}$$

Then

$$d_k^{T*} = \frac{b_{T-k+1}}{\sum_{i=1}^{T} b_i}, \quad k = 1, \ldots, T \quad \text{and} \quad m_T = \frac{\sum_{i=1}^{T} b_i}{\sum_{i=1}^{T} b_i g^{i-T}}$$

(The depreciation rate d_k^{T*} is non-negative since $\lambda > g > 1/\lambda > 0$, and this implies $b_k > 0$.) □

NOTES

1 Ijiri and Kaplan (1969, 1970) study depreciation in a setting in which cash flows are known but there is uncertainty regarding the asset's useful life.

2 A Kalman filter is an important development in forecasting and filtering. It is used when there is a need to separate signals from noise. OLS can be viewed as a special case of a Kalman filter. Examples abound. Airplanes fly on automatic control while a Kalman filter fits speed and position to the laws of motion. Kalman filters are used in communication problems in which the medium (e.g., the troposphere for radar, the ocean for sonar signals) acts as a potential source of noise.

3 '[Fibonacci numbers turn] up in a most fantastic variety of applications, and deserves a book of its own. Thorns and leaves grow in a spiral pattern, and on the hawthorn or apple or oak you find five [F_5] growths for every two [F_3] turns around the stem. The pear tree has eight [F_6] for every three turns [F_4] and the willow is even more complicated, 13 growths [F_7] for every five spirals [F_5]. The champion seems to be a sunflower of Daniel T. O'Connell (*Scientific American*, November, 1951) whose seeds chose an almost unbelievable ratio of $F_{12}/F_{13} = 144/233$' (Strang, 1988: 263). There is, in fact, a mathematical journal, *The Fibonacci Quarterly*, devoted entirely to Fibonacci numbers.

4 We assume each investment has zero salvage value. An alternative is to view I_t as being net of salvage value.

5 We use the definition of steady state as in Hatfield (1971/1927: 140–1). In steady state, the number of assets a firm possesses stays constant. In each period, one asset is fully depreciated and one new asset is purchased.

6 We thank the referees for suggesting this issue.

7 As n gets large, the ratio of successive Fibonacci numbers F_{n+1}/F_n approaches $(\sqrt{5}+1)/2$, known as the golden ratio. Our BLU weights are of the form F_i/F_{i+1} and, hence, they approach the inverse of the golden ratio.

8 In the limit, the declining balance method fully depreciates each asset.

9 See Demski (1994) for examples in which accounting information is used for statistical estimation in valuation (Chapter 13) and control (Chapter 19) settings.

10 Although there is no notion of matching in our model, our view of depreciation as an estimate of future expenditures (since current period's depreciation is also BLU for next period's investment) can be viewed as derived from matching. If depreciation tells us about the resources consumed in generating past revenues, it presumably also tells us about the resources needed to generate future revenues.

11 Obviously, *ex ante* the authors did not conjecture the closed-form expression for d_k^{I*}. A constructive proof – one that derives d_k^{I*} – uses a theorem for solving recursive equations (see, for example, Theorem 4.10, Niven *et al.*, 1991: 199). Interested readers can e-mail arya@cob.osu.edu for details of such a proof.

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