On the Role of Accruals

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Introduction

Accruals are pervasive in accounting practice. Further, accrual accounting competes with other information providers for resources. Yet much of accounting research criticizes their role or at least people's perceptions of accruals (for example, Sloan [1996], Ali, Hwang, and Trombley [2000], Collins and Hribar [2000], Bradshaw, Richardson, and Sloan [2001], Dechow and Dichev [2002], Hribar and Collins [2002], McNichols [2002], Thomas and Zhang [2002], Fairfield, Whisenant, and Yohn [2003], Barth and Hutton [2004], Desai, Rajgopal, and Venkatachalam [2004], and Zhang [2005]). In contrast, this paper explores positive valuation and evaluation roles of accruals in a simple, yet dynamic setting.

We show that accruals can summarize the information content regarding expected cash flows from the past cash flow history. This is similar in spirit to Arya, Fellingham, Glover, and Schroeder [2002]. In addition, we show in a moral hazard setting that the foregoing accrual statistic can be combined with current cash flows and non-accounting contractible information to efficiently (subject to LEN model restrictions[1]) supply incentives to agents. The LEN (linear exponential normal) model application is similar to Arya, Fellingham, and Schroeder [2004]. It is not surprising that accruals can serve as statistics for valuation or evaluation, rather the striking contribution here is that the same accrual statistic can serve both purposes without loss of efficiency.

An implication of the accruals to be described in more detail later is that they provide the same statistical information as fair value accounting but under weaker conditions. Suppose fair value accounting is applied to operating assets so that economic value V_t is readily at hand. From a statistical sufficiency perspective for the mean of cash flows, users are indifferent between the data

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[1] See Holmstrom and Milgrom [1987], for details on the strengths and limitations of the LEN (linear exponential normal) model.

pair current cash flow combined with previous period's value, and current cash flow combined with previous period's accruals.

The remainder of the paper is organized as follows. Section two develops the model, section three provides the results of the paper (two propositions and three corollaries; all proofs are in the appendix). Section four discusses connections with the extant empirical literature on accruals, and section five concludes the paper.

Model

The data generating process (DGP) is as follows. Period t cash flows (excluding the agent's compensation s) includes a permanent component m_t that derives from productive capital, the agent's contribution a_t, and a stochastic error e_t.

$$cf_t = m_t + a_t + e_t$$

The permanent component (mean) is subject to stochastic shocks.

$$m_t = g \ m_{t-1} + \epsilon_t$$

where m_0 is common knowledge, g is a deterministic growth factor, and stochastic shock ϵ_t. In addition, there exists contractible, non-accounting information that is informative of the agent's action a_t with noise μ_t.

$$y_t = a_t + \mu_t$$

The errors, e, ϵ, and μ are jointly normally distributed with mean zero and variance-covariance matrix Σ.

The agent has reservation wage RW and is evaluated subject to moral hazard. The agent's action is binary $a \in \{a_H, a_L\}$, $a_H > a_L$, with personal cost $c(a), c(a_H) > c(a_L)$, and the agent's preferences for payments s and actions are CARA $U(s,a) = -exp\{-r[s - c(a)]\}$. Payments are linear in performance measures w (with weights γ_t) plus flat wage $\delta_t, s_t = \delta_t + \gamma_t^T w$.

The valuation role of accruals is to summarize next period's unknown expected cash flow m_{t+1} based on the history of cash flows through time t. The incentive-induced equilibrium agent action a_t^* is effectively known for valuation purposes. Hence, the observable cash flow history at time t is $\{cf_1 - a_1^*, cf_2 - a_2^*, \ldots, cf_t - a_t^*\}$.

For the case $\Sigma = D$ where D is a diagonal matrix comprised of $\sigma_e^2, \sigma_\epsilon^2$, and σ_μ^2 (appropriately aligned), the following OLS regression identifies the most efficient usage of the past cash flow history. The current accrual equals the estimate of the current mean of cash flows scaled by g^{t-1} – the last element of the vector

$accruals_t = \frac{1}{g^{t-1}}\widehat{m}_t$ where

$$\widehat{m}_t = (H^T H)^{-1} H^T z,$$

$$H = \begin{bmatrix} -\nu & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \nu g & -\nu & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \nu g & -\nu \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, z = \begin{bmatrix} -\nu g\, m_0 \\ cf_1 - a_1^* \\ 0 \\ cf_2 - a_2^* \\ \vdots \\ 0 \\ cf_t - a_t^* \end{bmatrix}, \text{ and } \nu = \frac{\sigma_e}{\sigma_\varepsilon}.$$

On the other hand, the evaluation role of accruals must regard a_t as unobservable while previous actions of this or other agents are at the incentive-induced equilibrium action a^*, and all observables are potentially (conditionally) informative: $\{cf_1 - a_1^*, cf_2 - a_2^*, \ldots, cf_t\}$, and $\{y_1 - a_1^*, y_2 - a_2^*, \ldots, y_t\}$.[2]

For the case $\Sigma = D$, the most efficient linear contract can be found by determining the incentive portion of compensation via OLS and then plugging a constant δ to satisfy individual rationality.[3] The (linear) incentive payments are equal to the OLS estimator, the final element of $\widehat{a}_t$, multiplied by $\Delta = \frac{c(a_H) - c(a_L)}{a_H - a_L}$, $\gamma_t = \Delta\,\widehat{a}_t$[4] where

$$\widehat{a}_t = (H_a^T H_a)^{-1} H_a^T w,$$

$$H_a = \begin{bmatrix} -\nu & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \nu g & -\nu & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \nu g & -\nu & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & \phi \end{bmatrix}, w = \begin{bmatrix} -\nu g\, m_0 \\ cf_1 - a_1^* \\ 0 \\ cf_2 - a_2^* \\ \vdots \\ 0 \\ cf_t \\ \phi y_t \end{bmatrix}, \text{ and } \phi = \frac{\sigma_e}{\sigma_\varepsilon}.$$

Further, the variance of the incentive payments equals the last row, column element of $\Delta^2 (H_a^T H_a)^{-1}\sigma_e^2$.

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[2] For the case $\Delta = D$, past y's are uninformative of the current period's act.

[3] Individual rationality is satisfied if
$\delta = RW - \{E[incentivepayments|a] - \frac{1}{2}rVar[s] - c(a)\}.$

[4] The nuisance parameters (the initial $2t$ elements of $\widehat{a}_t$) could be avoided if one employs GLS in place of OLS.

Results

The valuation role of accruals is synthesized in proposition 1. Accruals can supply a sufficient summary of the cash flow history for the cash flow mean.[5] Next, we analyze accruals where the DGP involves unequal error variances and deterministic (as well as stochastic) growth in cash flows.

Growth with unequal variances

Consider a data generating process with $\Sigma = D$ involving unequal variances between e, ϵ, and μ, and a deterministic (as well as stochastic) growth element g. Let

$$m_t = g \; m_{t-1} + \epsilon_t, \nu = \frac{\sigma_e}{\sigma_\epsilon}, \text{ and } \phi = \frac{\sigma_e}{\sigma_\mu}. \text{ Also, } B = \begin{bmatrix} 1+\nu^2 & \nu^2 \\ g^2 & g^2\nu^2 \end{bmatrix} =$$

$S\Lambda S^{-1}$ where

$$\Lambda = \begin{bmatrix} \frac{1+\nu^2+g^2\nu^2-\sqrt{(1+\nu^2+g^2\nu^2)^2-4g^2\nu^4}}{2} & 0 \\ 0 & \frac{1+\nu^2+g^2\nu^2+\sqrt{(1+\nu^2+g^2\nu^2)^2-4g^2\nu^4}}{2} \end{bmatrix}$$

and

$$S = \begin{bmatrix} \frac{1+\nu^2-g^2\nu^2-\sqrt{(1+\nu^2+g^2\nu^2)^2-4g^2\nu^4}}{2g^2} & \frac{1+\nu^2-g^2\nu^2+\sqrt{(1+\nu^2+g^2\nu^2)^2-4g^2\nu^4}}{2g^2} \\ 1 & 1 \end{bmatrix}.$$

Now, define the difference equations by

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = B^t \begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = S\Lambda^t S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Accruals as a valuation statistic

The primary result involving accruals as a valuation statistic is presented in proposition 1.

Proposition 1 *Let $m_t = g \; m_{t-1} + e_t$, $\Sigma = D$, and $\nu = \frac{\sigma_e}{\sigma_\epsilon}$. Then, accruals$_{t-1}$ and cf_t are, collectively, sufficient statistics for the mean of cash flows m_t based on the history of cash flows and $g^{t-1}accruals_t$ is an efficient statistic for m_t*

$$\begin{aligned} [\widehat{m}_t | cf_1, ..., cf_t] &= g^{t-1}accruals_t \\ &= \frac{1}{den_t} \left\{ \frac{num_t}{g^2}(cf_t - a_t^*) + g^{t-1}\nu^2 den_{t-1} accruals_{t-1} \right\} \end{aligned}$$

where accruals$_0 = m_0$, and $\begin{bmatrix} den_t \\ num_t \end{bmatrix} = B^t \begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = S\Lambda^t S \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The variance of accruals is equal to the variance of the estimate of the mean of cash

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[5] As the agent's equilibrium contribution a^* is known, expected cash flow for the current period is estimated by $\widehat{m}_t + a_t^*$ and next period's expected cash flow is predicted by $g \; \widehat{m}_t + a_{t+1}^*$.

flows multiplied by $g^{2(t-1)}$; the variance of the estimate of the mean of cash flows equals the coefficient on current cash flow multiplied by σ_e^2, $Var\left[\widehat{m}_t\right] = \frac{num_t}{den_t g^2}\sigma_e^2$.

To appreciate the tidiness property of accruals in this setting it is instructive to consider the weight placed on the most recent cash flow as the number of periods becomes large. This limiting result is expressed in corollary 1.

Corollary 1 *As t becomes large, the weight on current cash flows for the efficient estimator of the mean of cash flows approaches*

$$\frac{2}{1 + (1 - g^2)\nu^2 + \sqrt{\left(1 + (1 + g^2)\nu^2\right)^2 + 4g^2\nu^4}}$$

and the variance of the estimate approaches

$$\frac{2}{1 + (1 - g^2)\nu^2 + \sqrt{\left(1 + (1 + g^2)\nu^2\right)^2 + 4g^2\nu^4}}\sigma_e^2.$$

Tidy accruals

Accruals, as identified above, are tidy in the sense that each period's cash flow is ultimately recognized in accounting income or remains as a "permanent" amount on the balance sheet.[6] This permanent balance is approximately

$$\sum_{t=1}^{k-1} cf_t \left[1 - \frac{num_t}{num_k} - num_t \sum_{n=t}^{k-1} \frac{g^{n-t-2}\nu^{2(n-1)}}{g^{n-1}den_n}\right]$$

where k is the first period where $\frac{num_t}{g^2 den_t}$ is well approximated by the asymptotic rate identified in corollary 1 and the estimate of expected cash flow $\widehat{m}_t$ is identified from tidy accruals as $g^{t-1}accruals_t$.[7] In the benchmark case, this balance reduces to

$$\sum_{t=1}^{k-1} cf_t \left[1 - \frac{F_{2t}}{F_{2k}} - F_{2t} \sum_{n=t}^{k-1} \frac{1}{F_{2n+1}}\right]$$

where the estimate of expected cash flow $\widehat{m}_t$ is equal to tidy $accruals_t$.

Performance evaluation

In a moral hazard setting, the incentive portion of the LEN contract based on cash flow and other monitoring information history is identified in proposition 2. Incentive payments depend only on two realizations: unexpected cash flow

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[6] The permanent balance is of course settled up on dissolution of the firm.

[7] Cash flows beginning with period k and after are fully accrued as the asymptotic rate effectively applies each period. Hence, a convergent geometric series is formed that sums to one. On the other hand, the permanent balance arises as a result of the influence of the common knowledge initial expected cash flow m_0.

and other monitoring information for period t. Unexpected cash flow at time t is

$$
\begin{aligned}
cf_t - E[cf_t|cf_1, \ldots, cf_{t-1}] &= cf_t - g^{t-1}accruals_{t-1} \\
&= cf_t - \widehat{m}_{t-1} \\
&= cf_t - [\widehat{m}_t|cf_1, \ldots, cf_{t-1}].
\end{aligned}
$$

As a result, sequential spot contracting with replacement agents has a particularly streamlined form. Accounting accruals supply a convenient and sufficient summary of the cash flow history for the cash flow mean. Hence, the combination of last period's accruals with current cash flow yields the pivotal unexpected cash flow variable.

Proposition 2 *Let $m_t = g\, m_{t-1} + e_t$, $\Sigma = D$, $\nu = \frac{\sigma_e}{\sigma_\epsilon}$, and $\phi = \frac{\sigma_e}{\sigma_\mu}$. Then, $accruals_{t-1}$, cf_t, and y_t, collectively, are sufficient statistics for evaluating the agent with incentive payments given by*

$$
\gamma_t = \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g^{t-1}accruals_{t-1} \right) \right]
$$

and variance of payments equal to

$$
Var[\gamma_t^T w] = \Delta^2 \frac{den_t}{\nu^2 den_{t-1} + \phi^2 den_t} \sigma_e^2
$$

where $accruals_{t-1}$ and den_t are as defined in proposition 1, and $\Delta = \frac{c(a_H) - c(a_L)}{aH - aL}$.

Benchmark case

Suppose $\Sigma = \sigma_e^2 I$ ($\nu = \phi = 1$) and $g = 1$. This benchmark case highlights the key informational structure in the data. Corollary 2 identifies the linear combination of current cash flows and last period's accruals employed to estimate the current cash flow mean conditional on cash flow history for this benchmark case.

Corollary 2 *For the benchmark case $\Sigma = \sigma_e^2 I$ ($\nu = \phi = 1$) and $g = 1$, accruals at time t are an efficient summary of past cash flow history for the cash flow mean if*

$$
\begin{aligned}
[\widehat{m}_t|cf_1, ..., cf_t] &= accruals_t \\
&= \frac{F_{2t}}{F_{2t+1}} (cf_t - a_t^*) + \frac{F_{2t-1}}{F_{2t+1}} accruals_{t-1}
\end{aligned}
$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (the Fibonacci series), and the sequence is initialized with $accruals_0 = m_0$ (common knowledge mean beliefs). Then, variance of accruals equals $Var[\widehat{m}_t] = \frac{F_{2t}}{F_{2t+1}} \sigma_e^2$.

For the benchmark case, the evaluation role of accruals is synthesized in corollary 3.

Corollary 3 *For the benchmark case* $\Sigma = \sigma_e^2 I$ $(\nu = \phi = 1)$ *and* $g = 1$, $accruals_{t-1}$, cf_t, *and* y_t *are, collectively, sufficient statistics for evaluating the agent with incentive payments given by*

$$\gamma_t = \Delta \left\{ \frac{F_{2t+1}}{L_{2t}} y_t + \frac{F_{2t-1}}{L_{2t}} \left(cf_t - accruals_{t-1} \right) \right\}$$

and variance of payments equals $\Delta^2 \frac{F_{2t+1}}{L_{2t}} \sigma_e^2$ *where* $accruals_{t-1}$ *is as defined in corollary 2,* $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$ *(the Lucas series), and* $\Delta = \frac{c(a_H) - c(a_L)}{a_H - a_L}$.[8]

Discussion

The extant empirical literature employs variations on reduced form models of accruals based on the initiatives of Jones [1991] and Dechow and Dichev [2002]. As is well known, reduced form models can be quite useful for exploring correlation in the data but provide limited help for identifying structural relations in the data generating process (DGP).[9] As Dechow and Dichev's approach as well as the DGP employed here are based on a firm's cash flow history, the remainder of this section develops connections between the two approaches.

Dechow and Dichev's empirical model of accruals is

$$accruals_t = \varpi + \omega_{-1} cf_{t-1} + \omega_0 cf_t + \omega_{+1} cf_{t+1} + \xi$$

The DGP employed in this paper yields accruals as a sufficient statistic for expected cash flows

$$accruals_t = \beta_0 + \beta_1 cf_1 + \beta_2 cf_2 + \cdots + \beta_t cf_t + \vartheta$$

where cf_t is assumed to be adjusted for the equilibrium contribution of the agent a_t^* everywhere (and the common knowledge mean m_0 at time 0 is subsumed in the intercept).

For the DGP employed throughout this paper, correlation inferred from the data is likely to produce an empirically-identified one-period lead-lag cash flow structure of accruals (developed below). Further, current cash flow is likely to have the most prominent correlation with accruals followed by one-period lead/lag cash flow. This, of course, is precisely the empirical structure initially identified by Dechow and Dichev and subsequently affirmed by numerous others.

Now, consider the correspondence between the accruals produced by the

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[8] The Lucas and Fibonacci series are related by $L_n = F_{n-1} + F_{n+1}$, for $n = 1, 2, \dots$.

[9] See Heckman [2001] and Heckman and Vytlacil [2005] for broad-based discussions of the benefits and limitations of reduced form models for policy evaluation.

DGP to the lead-lag structure identified empirically by Dechow and Dichev.

$$\beta_0 + \beta_1 cf_1 + \beta_2 cf_2 + \cdots + \beta_t cf_t + \vartheta$$
$$= \varpi + \omega_{-1} cf_{t-1} + \omega_0 cf_t + \omega_{+1} cf_{t+1} + \xi$$

The focus is on the plim of ω_0, ω_{-1}, ω_{+1}. The plim of ω does not equal its β counterpart for two reasons. First, accruals are designed to be contractible performance measures and accordingly include no lead terms (this is relaxed below). Second, there is an omitted, correlated variable problem with the simple one-period lag structure. Recall the BLU estimator (for expected cash flow) employs the entire history of cash flows while the empirical models are typically truncated at one lag. This produces a discrepancy between the plim of ω and its corresponding β.

For the benchmark case ($\nu = 1$), the plim of ω and its corresponding β are tabulated below for $t = 3$ or 4.

	$t = 3$	$t = 4$
ω_{-1}	$\frac{68}{273}$	$\frac{45018}{169099}$
β_{t-1}	$\frac{3}{13}$	$\frac{8}{34}$
$\omega_{-1} - \beta_{t-1}$	0.0183	0.0309
ω_0	$\frac{170}{273}$	$\frac{213071}{338198}$
β_t	$\frac{8}{13}$	$\frac{21}{34}$
$\omega_0 - \beta_t$	0.0073	0.0124
ω_{+1}	$\frac{1}{273}$	$\frac{1046}{169099}$
β_{t+1}	0	0
$\omega_{+1} - \beta_{t+1}$	0.0037	0.0062

Discrepancies between the plim of ω and β
based on cash flow history through period t

Now, suppose one ignores the performance evaluation role of accruals[10] and includes period $t + 1$ cash flows in the period t accrual statistic for m_t.[11] This eliminates the first problem discussed above and only the omitted variable problem arising from the missing past history produces a discrepancy between the plim of ω and its corresponding β. For the benchmark case ($g = \nu = 1$), the plim of ω and its corresponding β are tabulated below for $t = 3$ or 4. As expected, the discrepancy between the plim of ω and its corresponding β is mitigated when the first source is eliminated.

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[10] Another interpretation is that the principal wishes to employ subjective as well as objective performance measures (see Rajan and Reichelstein [2006]). Then, perhaps, the cash flow history utilized by accruals includes future as well as past realizations to summarize expectations of future cash flows.

[11] There is potential for confusion regarding which mean is the one of interest, m_t or m_{t+1}? We examine the updating of m_t based on the cash flow history $\{cf_1, cf_2, \ldots, cf_{t+1}\}$ to focus on period t.

	$t = 3$	$t = 4$
ω_{-1}	$\frac{4}{21}$	$\frac{6}{29}$
β_{t-1}	$\frac{3}{17}$	$\frac{16}{89}$
$\omega_{-1} - \beta_{t-1}$	0.0140	0.0271
ω_0	$\frac{4}{21}$	$\frac{14}{29}$
β_t	$\frac{3}{17}$	$\frac{42}{89}$
$\omega_0 - \beta_t$	0.0056	0.0108
ω_{+1}	$\frac{4}{21}$	$\frac{7}{29}$
β_{t+1}	$\frac{3}{17}$	$\frac{21}{89}$
$\omega_{+1} - \beta_{t+1}$	0.0028	0.0054

Discrepancies between the plim of ω and β
based on cash flow history through period $t + 1$

Continuing the theme that accruals are a sufficient statistic for expected cash flows given a firm's cash flow history, the above discrepancies are model specification errors. Such specification errors confound attempts to identify accrual attributes such as discretionary, non-discretionary, etc.[12]

Conclusions

A positive view of accruals is outlined in this paper. Accruals combined with current cash flow can serve as sufficient statistics of the cash flow history for the mean of cash flows. Further, in a moral hazard setting accruals can be combined with current cash flow and other monitoring information to efficiently evaluate replacement agents via sequential spot contracts. Notably, the same accrual statistic serves both valuation and evaluation purposes.

An implication of the accruals described above is that they provide the same statistical information as fair value accounting but under weaker conditions than required for fair value accounting. Suppose fair value accounting is applied to operating assets so that economic value V_t and economic income are readily at hand. From a statistical sufficiency perspective, users are indifferent between $\{cf_t - a_t^*, V_{t-1}\}$ and $\{cf_t - a_t^*, accruals_{t-1}\}$. Each pair of data is a sufficient summary of the cash flow history for expected cash flows.

While the accruals measure discussed here does not directly serve as a model for empirical work, it does serve as a reminder that reduced-form models likely contain specification errors that limit their ability to help identify structural attributes of accruals.

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[12] It should be noted that these attributes are not well-defined in the context of this paper.

Appendix

Proposition 1. *Let $m_t = g\,m_{t-1} + e_t$, $\Sigma = D$, and $\nu = \frac{\sigma_e}{\sigma_\epsilon}$. Then, $accruals_{t-1}$ and cf_t are, collectively, sufficient statistics for the mean of cash flows m_t based on the history of cash flows and $g^{t-1} accruals_t$ is an efficient statistic for m_t*

$$
\begin{aligned}
[\widehat{m}_t | cf_1, ..., cf_t] &= g^{t-1} accruals_t \\
&= \frac{1}{den_t}\left\{ \frac{num_t}{g^2}(cf_t - a_t^*) + g^{t-1}\nu^2 den_{t-1} accruals_{t-1} \right\}
\end{aligned}
$$

where $accruals_0 = m_0$, and $\begin{bmatrix} den_t \\ num_t \end{bmatrix} = B^t \begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = S\Lambda^t S \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The variance of accruals is equal to the variance of the estimate of the mean of cash flows multiplied by $g^{2(t-1)}$; the variance of the estimate of the mean of cash flows equals the coefficient on current cash flow multiplied by σ_e^2, $Var[\widehat{m}_t] = \frac{num_t}{den_t g^2}\sigma_e^2$.
Proof. Outline of the proof:

1. Since the data are multivariate normally distributed, BLU estimation is efficient (achieves the Cramer-Rao lower bound amongst consistent estimators; see Greene [1997], p. 300-302).

2. BLU estimation is written as a recursive least squares exercise (see Strang [1986], p. 146-148).

3. The proof is completed by induction. That is, the difference equation solution is shown, by induction, to be equivalent to the recursive least squares estimator. A key step is showing that the information matrix $\Im$ and its inverse can be derived in recursive fashion via LDL^T factorization (i.e., $D^{-1}L^{-1}\Im = L^T$).

Recursive least squares. Let $H_1 = \begin{bmatrix} -\nu \\ 1 \end{bmatrix}$ (a 2 by 1 matrix), $H_2 = \begin{bmatrix} g\nu & -\nu \\ 0 & 1 \end{bmatrix}$ (a 2 by 2 matrix), $H_t = \begin{bmatrix} 0 & \cdots & 0 & g\nu & -\nu \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$ (a 2 by t matrix with $t-2$ leading columns of zeroes), $z_1 = \begin{bmatrix} -g\nu m_0 \\ cf_1 - a_1^* \end{bmatrix}$, $z_2 = \begin{bmatrix} 0 \\ cf_2 - a_2^* \end{bmatrix}$, and $z_t = \begin{bmatrix} 0 \\ cf_t - a_t^* \end{bmatrix}$. The information matrix for a t-period cash flow history is

$$
\begin{aligned}
\Im_t &= \Im_{t-1}^a + H_t^T H_t \\
&= \begin{bmatrix}
1+\nu^2+g^2\nu^2 & -g\nu^2 & 0 & \cdots & 0 \\
-g\nu^2 & 1+\nu^2+g^2\nu^2 & -g\nu^2 & \ddots & \vdots \\
0 & -g\nu^2 & \ddots & -g\nu^2 & 0 \\
\vdots & \ddots & -g\nu^2 & 1+\nu^2+g^2\nu^2 & -g\nu^2 \\
0 & \cdots & 0 & -g\nu^2 & 1+\nu^2
\end{bmatrix},
\end{aligned}
$$

a symmetric tri-diagonal matrix, where $\Im_{t-1}^a$ is $\Im_{t-1}$ augmented with a row and column of zeroes to conform with $\Im_t$. For instance, $\Im_1 = \begin{bmatrix} 1 + \nu^2 \end{bmatrix}$ and $\Im_1^a = \begin{bmatrix} 1 + \nu^2 & 0 \\ 0 & 0 \end{bmatrix}$. The estimate of the mean of cash flows is derived recursively as

$$b_t = b_{t-1}^a + k_t \left(z_t - H_t b_{t-1}^a \right)$$

for $t > 1$ where $k_t = \Im_t^{-1} H_t^T$, the gain matrix, and b_{t-1}^a is b_{t-1} augmented with a zero to conform with b_t. The best linear unbiased estimate of the current mean is the last element in the vector b_t and its variance is the last row-column element of $\Im_t^{-1}$ multiplied by σ_e^2.

Difference equations. The difference equations are

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = \begin{bmatrix} 1 + \nu^2 & \nu^2 \\ g^2 & g^2\nu^2 \end{bmatrix} \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}$$

with $\begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The difference equations estimator for the current mean of cash flows and its variance are

$$
\begin{aligned}
\widehat{m}_t &= \frac{1}{den_t} \left(\frac{num_t}{g^2} (cf_t - a_t^*) + g\nu^2 den_{t-1}\widehat{m}_{t-1} \right) \\
&= g^{t-1} \ accruals_t = \frac{1}{den_t} \left(\frac{num_t}{g^2} (cf_t - a_t^*) + g^{t-1}\nu^2 den_{t-1} accruals_{t-1} \right)
\end{aligned}
$$

where $accruals_0 = m_0$, and

$$Var\left[\widehat{m}_t\right] = g^{2(t-1)} Var\left[accruals_t\right] = \sigma_e^2 \frac{num_t}{g^2 den_t}.$$

Induction steps. Assume

$$
\begin{aligned}
\widehat{m}_t &= \frac{1}{den_t} \left(\frac{num_t}{g^2} (cf_t - a_t^*) + g\nu^2 den_{t-1}\widehat{m}_{t-1} \right) \\
&= g^{t-1} \ accruals_t = \frac{1}{den_t} \left(\frac{num_t}{g^2} (cf_t - a_t^*) + g^{t-1}\nu^2 den_{t-1} accruals_{t-1} \right) \\
&= \left[b_{t-1}^a + k_t \left(z_t - H_t b_{t-1}^a \right) \right] [t]
\end{aligned}
$$

and

$$Var\left[\widehat{m}_t\right] = g^{2(t-1)} Var\left[accruals_t\right] = Var\left[b_t\right][t, t]$$

where $[t]$ $([t, t])$ refers to element t (t, t) in the vector (matrix). The above is clearly true for the base case, $t = 1$ and $t = 2$. Now, show

$$
\begin{aligned}
\widehat{m}_{t+1} &= \frac{1}{den_{t+1}} \left(\frac{num_{t+1}}{g^2} \left(cf_{t+1} - a_{t+1}^* \right) + g^t \nu^2 den_t accruals_t \right) \\
&= \left[b_t^a + k_{t+1} \left(z_{t+1} - H_{t+1} b_t^a \right) \right] [t+1].
\end{aligned}
$$

Recall $z_{t+1} = \begin{bmatrix} 0 \\ c_{t+1} - a_{t+1}^* \end{bmatrix}$ and $H_{t+1} = \begin{bmatrix} 0 & \cdots & 0 & g\nu & -\nu \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$. From LDL^T factorization of $\Im_{t+1}$ (recall $L^T = D^{-1}L^{-1}\Im$ where L^{-1} is simply products of matrices reflecting successive row eliminations - no row exchanges are involved due to the tri-diagonal structure and D^{-1} is the reciprocal of the diagonal elements remaining following eliminations) the last row of $\Im_{t+1}^{-1}$ is

$$\begin{bmatrix} \dfrac{g^{t-1}\nu^{2(t-1)}num_1}{g^2 den_{t+1}} & \cdots & \dfrac{g^2\nu^4 num_{t-1}}{g^2 den_{t+1}} & \dfrac{g\nu^2 num_t}{g^2 den_{t+1}} & \dfrac{num_{t+1}}{g^2 den_{t+1}} \end{bmatrix}.$$

This immediately identifies the variance associated with the estimator as the last term in $\Im_{t+1}^{-1}$ multiplied by the variance of cash flows, $\dfrac{num_{t+1}}{g^2 den_{t+1}}\sigma_e^2$. Hence, the difference equation and the recursive least squares variance estimators are equivalent.

Since $H_{t+1}^T z_{t+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ cf_{t+1} - a_{t+1}^* \end{bmatrix}$, the lead term on the RHS of the $[t+1]$

mean estimator is $\dfrac{num_{t+1}}{g^2 den_{t+1}}\left(cf_{t+1} - a_{t+1}^*\right)$ which is identical to the lead term on the left hand side (LHS). Similarly, the second term on the RHS (recall the focus is on element t, the last element of b_t^a is 0) is

$$[(I - k_{t+1}H_{t+1})b_t^a][t+1]$$

$$= \left[\left(I - \Im_{t+1}^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \vdots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & g^2\nu^2 & -g\nu^2 \\ 0 & \cdots & 0 & -g\nu^2 & 1+\nu^2 \end{bmatrix} \right) b_t^a \right][t+1]$$

$$= \left(\dfrac{-g^3\nu^4 num_t}{g^2 den_{t+1}} + \dfrac{g\nu^2 num_{t+1}}{g^2 den_{t+1}} \right)\widehat{m}_t$$

$$= \left(\dfrac{-g^3\nu^4 num_t + g\nu^2 num_{t+1}}{g^2 den_{t+1}} \right)g^{t-1}\ accruals_t.$$

The last couple of steps involves substitution of $\widehat{m}_t$ for $b_t^a[t+1]$ and $g^{t-1}accruals_t$ for $\widehat{m}_t$ on the right hand side (RHS). The difference equation relation, $num_{t+1} = g^2 den_t + g^2\nu^2 num_t$, implies

$$\dfrac{-g^3\nu^4 num_t + g\nu^2 num_{t+1}}{g^2 den_{t+1}}\widehat{m}_t = \dfrac{1}{den_{t+1}}g\nu^2 den_t\ \widehat{m}_t =$$

$$= \dfrac{1}{den_{t+1}}g^t\nu^2 den_t\ accruals_t$$

the second term on the LHS. This completes the induction steps. ∎

Corollary 1. *As t becomes large, the weight on current cash flows for the efficient estimator of the mean of cash flows approaches*

$$\frac{2}{1 + (1 - g^2)\,\nu^2 + \sqrt{\left(1 + (1 + g^2)\,\nu^2\right)^2 + 4g^2\nu^4}}$$

and the variance of the estimate approaches

$$\frac{2}{1 + (1 - g^2)\,\nu^2 + \sqrt{\left(1 + (1 + g^2)\,\nu^2\right)^2 + 4g^2\nu^4}}\sigma_e^2.$$

Proof. The difference equations

$$
\begin{aligned}
\begin{bmatrix} den_t \\ num_t \end{bmatrix} &= S\Lambda^t S^{-1}\begin{bmatrix} den_0 \\ num_0 \end{bmatrix} \\
&= S\Lambda^t S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = S\Lambda^t c
\end{aligned}
$$

imply

$$
c = S^{-1}\begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = \begin{bmatrix} \frac{-g^2}{1+(1+g^2)\nu^2+\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}} \\ \frac{g^2}{1+(1+g^2)\nu^2+\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}} \end{bmatrix}.
$$

Thus,

$$
\begin{bmatrix} den_t \\ num_t \end{bmatrix} = S\begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} c
$$

$$
= \begin{bmatrix} \frac{\lambda_2^t\left(1+(1-g^2)\nu^2+\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}\right)-\lambda_1^t\left(1+(1-g^2)\nu^2-\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}\right)}{2\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}} \\ \frac{g^2\left(\lambda_2^t-\lambda_1^t\right)}{\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}} \end{bmatrix}.
$$

Since λ_2 is larger than λ_1, λ_1^t contributes negligibly to $\begin{bmatrix} den_t \\ num_t \end{bmatrix}$ for arbitrarily large t. Hence,

$$
\lim_{t\to\infty}\frac{num_t}{g^2 den_t} = \frac{2}{1 + (1 - g^2)\,\nu^2 + \sqrt{\left(1 + (1 + g^2)\,\nu^2\right)^2 - 4g^2\nu^4}}.
$$

From proposition 1, the variance of the estimator for expected cash flow is

13

$\frac{num_t}{g^2 den_t}\sigma_e^2$. Since

$$\lim_{t\to\infty} \frac{num_t}{g^2 den_t} = \frac{2}{1 + (1-g^2)\nu^2 + \sqrt{(1+(1+g^2)\nu^2)^2 - 4g^2\nu^4}}.$$

the asymptotic variance is

$$\frac{2}{1 + (1-g^2)\nu^2 + \sqrt{(1+(1+g^2)\nu^2)^2 - 4g^2\nu^4}}\sigma_e^2.$$

This completes the asymptotic case. ∎

Proposition 2. *Let* $m_t = g\, m_{t-1} + e_t$, $\Sigma = D$, $\nu = \frac{\sigma_e}{\sigma_\epsilon}$, *and* $\phi = \frac{\sigma_e}{\sigma_\mu}$. *Then,* $accruals_{t-1}$, cf_t, *and* y_t, *collectively, are sufficient statistics for evaluating the agent with incentive payments given by*

$$\gamma_t = \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1}\left(cf_t - g^{t-1}accruals_{t-1}\right)\right]$$

and variance of payments equal to

$$Var[\gamma_t^T w] = \Delta^2 \frac{den_t}{\nu^2 den_{t-1} + \phi^2 den_t}\sigma_e^2$$

where $accruals_{t-1}$ *and* den_t *are as defined in proposition 1, and* $\Delta = \frac{c(a_H)-c(a_L)}{aH-aL}$.
Proof. Outline of the proof:

1. Show that the "best" linear contract is equivalent to the BLU estimator of the agent's current act rescaled by the agent's marginal cost of the act.

2. The BLU estimator is written as a recursive least squares exercise (see Strang [1986], p. 146-148).

3. The proof is completed by induction. That is, the difference equation solution is shown, by induction, to be equivalent to the recursive least squares estimator. Again, a key step involves showing that the information matrix $\Im_a$ and its inverse can be derived in recursive fashion via LDL^T factorization (i.e., $D^{-1}L^{-1}\Im_a = L^T$).

"Best" linear contracts. The program associated with the optimal a_H-inducing LEN contract written in certainty equivalent form is

$$\underset{\delta,\gamma}{Min}\ \ \delta + E\left[\gamma^T w | a_H\right]$$

subject to

$$\delta + E\left[\gamma^T w | a_H\right] - \frac{r}{2}Var\left[\gamma^T w\right] - c\left(a_H\right) \geq RW \tag{IR}$$

$$\delta + E\left[\gamma^T w | a_H\right] - \frac{r}{2} Var\left[\gamma^T w\right] - c\left(a_H\right)$$

$$\geq \delta + E\left[\gamma^T w | a_L\right] - \frac{r}{2} Var\left[\gamma^T w\right] - c\left(a_L\right) \quad \text{(IC)}$$

As demonstrated in Arya, Fellingham, and Schroeder [2004], both IR and IC are binding and γ equals the BLU estimator of a based on the history w (the history of cash flows cf and other contractible information y) rescaled by the agent's marginal cost of the act $\Delta = \frac{c(a_H) - c(a_L)}{a_H - a_L}$. Since IC is binding,

$$\delta + E\left[\gamma^T w | a_H\right] - \frac{r}{2} Var\left[\gamma^T w\right] - \left(\delta + E\left[\gamma^T w | a_L\right] - \frac{r}{2} Var\left[\gamma^T w\right]\right)$$
$$= c\left(a_H\right) - c\left(a_L\right)$$

$$E\left[\gamma^T w | a_H\right] - E\left[\gamma^T w | a_L\right] = c\left(a_H\right) - c\left(a_L\right)$$

$$\gamma^T \left\{E\left[w | a_H\right] - E\left[w | a_L\right]\right\} = c\left(a_H\right) - c\left(a_L\right)$$

$$\left(a_H - a_L\right) \gamma^T \iota = c\left(a_H\right) - c\left(a_L\right)$$

where

$$w = \begin{bmatrix} cf_1 - m_0 - a_1^* \\ cf_2 - m_0 - a_2^* \\ \vdots \\ cf_t - m_0 \\ y_t \end{bmatrix}$$

and ι is a vector of zeroes except the last two elements are equal to one, and

$$\gamma^T \iota = \frac{c\left(a_H\right) - c\left(a_L\right)}{a_H - a_L}.$$

Notice, the sum of the last two elements of γ equals one, $\gamma^T \iota = 1$, is simply the unbiasedness condition associated with the variance minimizing estimator of a based on design matrix H_a. Hence, γ equals the BLU estimator of a rescaled by Δ, $\gamma_t = \Delta \hat{a}_t$. As δ is a free variable, IR can always be exactly satisfied by setting

$$\delta = RW - \left\{E\left[\gamma^T w | a_H\right] - \frac{r}{2} Var\left[\gamma^T w\right] - c\left(a_H\right)\right\}.$$

Recursive least squares. H_t remains as defined in the proof of proposition 1. Let $H_{a1} = \begin{bmatrix} -\nu & 0 \\ 1 & 1 \\ 0 & \phi \end{bmatrix}$ (a 3 by 2 matrix), $H_{a2} = \begin{bmatrix} g\nu & -\nu & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \phi \end{bmatrix}$ (a 3 by 3 matrix), $H_{at} = \begin{bmatrix} 0 & \cdots & 0 & g\nu & -\nu & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \phi \end{bmatrix}$ (a 3 by $t+1$ matrix with leading zeroes), $\widetilde{w}_1 = \begin{bmatrix} -g\nu m_0 \\ cf_1 \\ y_1 \end{bmatrix}$, $\widetilde{w}_2 = \begin{bmatrix} 0 \\ cf_2 \\ y_2 \end{bmatrix}$, and $\widetilde{w}_t = \begin{bmatrix} 0 \\ cf_t \\ y_t \end{bmatrix}$. The information

matrix for a t-period cash flow and other monitoring information history is

$$\Im_{at} = \Im_{t-1}^{aa} + H_{at}^T H_{at}$$

$$= \begin{bmatrix} 1+\nu^2+g^2\nu^2 & -g\nu^2 & 0 & 0 & \cdots & 0 \\ -g\nu^2 & 1+\nu^2+g^2\nu^2 & -g\nu^2 & \ddots & \cdots & 0 \\ 0 & -g\nu^2 & \ddots & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & 1+\nu^2+g^2\nu^2 & -g\nu^2 & 0 \\ \vdots & \cdots & 0 & -g\nu^2 & 1+\nu^2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1+\phi^2 \end{bmatrix}$$

a symmetric tri-diagonal matrix where $\Im_{t-1}^{aa}$ is $\Im_{t-1}^a$ (the augmented information matrix employed to estimate the cash flow mean in proposition 1) augmented with an additional row and column of zeroes (i.e., the information matrix from proposition 1, $\Im_{t-1}$, is augmented with two columns of zeroes on the right and two rows of zeroes on the bottom). The recursive least squares estimator is

$$b_{at} = \left[b_{t-1}^{aa} + k_{at} \left(\widetilde{w}_t - H_{at} b_{t-1}^{aa} \right) \right]$$

for $t > 1$ where b_{t-1}^{aa} is b_{t-1} (the accruals estimator of m_{t-1} from proposition 1) augmented with two zeroes and $k_{at} = \Im_{at}^{-1} H_{at}^T$. The best linear unbiased estimate of the current act is the last element in the vector b_{at} and its variance is the last row-column element of $\Im_{at}^{-1}$ multiplied by σ_e^2. Notice, recursive least squares applied to the performance evaluation exercise utilizes the information matrix $\Im_{t-1}^{aa}$ (the information matrix employed in proposition 1 augmented with two trailing rows-columns of zeroes) and estimator b_{t-1}^{aa} (the accruals estimator of m_{t-1} from proposition 1 augmented with the two trailing zeroes). This accounts for the restriction on the parameters due to past actions already having been motivated in the past (i.e., past acts are at their equilibrium level a^*). Only the current portion of the design matrix H_{at} and the current observations w_t (in place of z_t) differ from the setup for accruals (in proposition 1).

Difference equations. The difference equations are

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = \begin{bmatrix} 1+\nu^2 & \nu^2 \\ g^2 & g^2\nu^2 \end{bmatrix} \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}$$

with $\begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The difference equations estimator for the linear incentive payments γ is

$$\begin{aligned} \gamma_t &= \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g\, \widehat{m}_{t-1} \right) \right] \\ &= \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g^{t-1}\, accruals_{t-1} \right) \right] \end{aligned}$$

and the variance of payments is

$$Var\left[\gamma^T w\right] = \Delta^2 \frac{den_t}{\nu^2 den_{t-1} + \phi^2 den_t}\sigma_e^2.$$

Induction steps. Assume

$$
\begin{aligned}
\gamma_t w &= \Delta\frac{1}{\nu^2 den_{t-1} + \phi^2 den_t}\left[\phi^2 den_t y_t + \nu^2 den_{t-1}\left(cf_t - g\ \widehat{m}_{t-1}\right)\right]\\
&= \Delta\frac{1}{\nu^2 den_{t-1} + \phi^2 den_t}\left[\phi^2 den_t y_t + \nu^2 den_{t-1}\left(cf_t - g^{t-1}\ accruals_{t-1}\right)\right]\\
&= \Delta\left[b_{t-1}^a + k_{at}\left(w_t - H_{at}b_{t-1}^a\right)\right][t+1]
\end{aligned}
$$

and

$$Var\left[\gamma^T w\right] = \Delta^2 Var\left[\widehat{a}_t\right][t+1, t+1]$$

where $[t+1]$ $([t+1, t+1])$ refers to element $t+1$ $(t+1, t+1)$ in the vector (matrix). The above is clearly true for the base case, $t = 1$ and $t = 2$. Now, show

$$
\begin{aligned}
&\Delta\frac{1}{\nu^2 den_t + \phi^2 den_{t+1}}\left[\phi^2 den_{t+1} y_{t+1} + \nu^2 den_t\left(cf_{t+1} - g\ \widehat{m}_t\right)\right]\\
=\ &\Delta\frac{1}{\nu^2 den_t + \phi^2 den_{t+1}}\left[\phi^2 den_{t+1} y_{t+1} + \nu^2 den_t\left(cf_{t+1} - g^t\ accruals_t\right)\right]\\
=\ &\Delta\left[b_t^a + k_{at+1}\left(\widetilde{w}_{t+1} - H_{at+1}b_t^a\right)\right][t+2].
\end{aligned}
$$

Recall $\widetilde{w}_{t+1} = \begin{bmatrix} 0 \\ cf_{t+1} \\ \phi y_{t+1} \end{bmatrix}$ and $H_{at+1} = \begin{bmatrix} 0 & \cdots & 0 & g\nu & \nu & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \phi \end{bmatrix}$. From

LDL^T factorization of $\Im_{at+1}$ (recall $L^T = D^{-1}L^{-1}\Im_a$ where L^{-1} is simply products of matrices reflecting successive row eliminations - no row exchanges are involved due to the tri-diagonal structure and D^{-1} is the reciprocal of the remaining elements remaining after eliminations) the last row of $\Im_{at+1}^{-1}$ is

$$\frac{1}{\nu^2 den_t + \phi^2 den_{t+1}}\begin{bmatrix} -g^{t-1}\nu^{2(t-1)}den_1 \\ \vdots \\ -g\nu^2\left(den_{t-1} + \nu^2 num_{t-1}\right) \\ -\left(den_t + \nu^2 num_t\right) \\ den_{t+1} \end{bmatrix}^T.^{13}$$

This immediately identifies the variance associated with the estimator as the last term in $\Im_{at+1}^{-1}$ multiplied by the product of the agent's marginal cost of the act squared and the variance of cash flows, $\Delta^2 \frac{den_{t+1}}{\nu^2 den_t + \phi^2 den_{t+1}}\sigma_e^2$. Hence, the difference equation and the recursive least squares variance of payments

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[13] Transposed due to space limitations.

estimators are equivalent.

Since $H_{at+1}^T \widetilde{w}_{t+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ cf_{t+1} \\ cf_{t+1} + y_{t+1} \end{bmatrix}$ and the difference equation implies

$den_{t+1} = \left(1 + \nu^2\right) den_t + \nu^2 num_t$, the lead term on the RHS is

$$\frac{den_{t+1}}{\nu^2 den_t + \phi^2 den_{t+1}} \left(y_{t+1} + cf_{t+1}\right) - \frac{den_t + \nu^2 num_t}{\nu^2 den_t + \phi^2 den_{t+1}} cf_{t+1}$$

$$= \frac{den_{t+1}}{\nu^2 den_t + \phi^2 den_{t+1}} y_{t+1} - \frac{\nu^2 den_t}{\nu^2 den_t + \phi^2 den_{t+1}} cf_{t+1}$$

which equals the initial expression on the LHS of the $[t+2]$ incentive payments. Similarly, the $\widehat{m}_t = g^{t-1} \ accruals_t$ term on the RHS (recall the focus is on element $t+2$) is

$$\left[(I - k_{at+1} H_{at+1}) b_t^a\right] [t+2]$$

$$= \left[\left(I - \Im_{at+1}^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & g^2\nu^2 & -g\nu^2 & 0 \\ 0 & \cdots & 0 & -g\nu^2 & 1+\nu^2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 1+\phi^2 \end{bmatrix}\right) b_t^a\right] [t+2]$$

$$= -\frac{g\nu^2 den_t}{\nu^2 den_t + \phi^2 den_{t+1}} \widehat{m}_t$$

$$= -\frac{g^t \nu^2 den_t}{\nu^2 den_t + \phi^2 den_{t+1}} accruals_t.$$

Combining terms and simplifying produces the result

$$\frac{1}{\nu^2 den_t + \phi^2 den_{t+1}} \left[\phi^2 den_{t+1} y_{t+1} + \nu^2 den_t \left(cf_{t+1} - g \ \widehat{m}_t\right)\right]$$

$$= \frac{1}{\nu^2 den_t + \phi^2 den_{t+1}} \left[\phi^2 den_{t+1} y_{t+1} + \nu^2 den_t \left(cf_{t+1} - g^t \ accruals_t\right)\right].$$

Finally, recall the estimator $\widehat{a}_t$ (the last element of b_{at}) rescaled by the agent's

marginal cost of the act identifies the "best" linear incentive payments

$$
\begin{aligned}
\gamma_t w &= \Delta \widehat{a}_t \\
&= \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g \ \widehat{m}_{t-1} \right) \right] \\
&= \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g^{t-1} \ accruals_{t-1} \right) \right].
\end{aligned}
$$

This completes the induction steps. ∎

Corollary 2. *For the benchmark case* $\Sigma = \sigma_e^2 I$ $(\nu = \phi = 1)$ *and* $g = 1$, *accruals at time* t *are an efficient summary of past cash flow history for the cash flow mean if*

$$
\begin{aligned}
[\widehat{m}_t | cf_1, ..., cf_t] &= accruals_t \\
&= \frac{F_{2t}}{F_{2t+1}} \left(cf_t - a_t^* \right) + \frac{F_{2t-1}}{F_{2t+1}} accruals_{t-1}
\end{aligned}
$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ *(the Fibonacci series), and the sequence is initialized with* $accruals_0 = m_0$ *(common knowledge mean beliefs). Then, variance of accruals equals* $Var \left[\widehat{m}_t \right] = \frac{F_{2t}}{F_{2t+1}} \sigma_e^2$.
Proof. Replace $g = \nu = 1$ in proposition 1. Hence,

$$
\left[\begin{array}{c} den_t \\ num_t \end{array} \right] = B \left[\begin{array}{c} den_{t-1} \\ num_{t-1} \end{array} \right]
$$

reduces to

$$
\left[\begin{array}{c} den_t \\ num_t \end{array} \right] = \left[\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} den_{t-1} \\ num_{t-1} \end{array} \right].
$$

Since

$$
\left[\begin{array}{c} F_{n+1} \\ F_n \end{array} \right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} F_n \\ F_{n-1} \end{array} \right]
$$

and

$$
\left[\begin{array}{c} F_{n+2} \\ F_{n+1} \end{array} \right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} F_n \\ F_{n-1} \end{array} \right] = \left[\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} F_n \\ F_{n-1} \end{array} \right],
$$

$$
den_t = F_{2t+1}, num_t = F_{2t}, den_{t-1} = F_{2t-1}, \text{and } num_{t-1} = F_{2t-2}.
$$

For $g = \nu = 1$, the above implies

$$
\widehat{m}_t = g^{t-1} accruals_t = \frac{1}{den_t} \left(\frac{num_t}{g^2} \left(cf_t - a_t^* \right) + g^{t-1} \nu^2 den_{t-1} accruals_{t-1} \right)
$$

reduces to

$$
\frac{F_{2t}}{F_{2t+1}} \left(cf_t - a_t^* \right) + \frac{F_{2t-1}}{F_{2t+1}} accruals_{t-1}
$$

19

and variance of accruals equals $\frac{F_{2t}}{F_{2t+1}}\sigma_e^2$. ∎

Corollary 3. *For the benchmark case* $\Sigma = \sigma_e^2 I$ $(\nu = \phi = 1)$ *and* $g = 1$, *accruals*$_{t-1}$, *cf*$_t$, *and* y_t *are, collectively, sufficient statistics for evaluating the agent with incentive payments given by*

$$\gamma_t = \Delta \left\{ \frac{F_{2t+1}}{L_{2t}} y_t + \frac{F_{2t-1}}{L_{2t}} \left(cf_t - accruals_{t-1} \right) \right\}$$

and variance of payments equals $\Delta^2 \frac{F_{2t+1}}{L_{2t}} \sigma_e^2$ *where accruals*$_{t-1}$ *is as defined in corollary 2 and* $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, *and* $L_1 = 1$ *(the Lucas series), and* $\Delta = \frac{c(a_H) - c(a_L)}{a_H - a_L}$.

Proof. Replace $g = \nu = \phi = 1$ in proposition 2. Hence,

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = B \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}$$

reduces to

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}.$$

Since

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

and

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

$den_t = F_{2t+1}$, $num_t = F_{2t}$, $den_{t-1} = F_{2t-1}$, $num_{t-1} = F_{2t-2}$, and $L_t = F_{t+1} + F_{t-1}$. For $g = \nu = \phi = 1$, the above implies

$$\gamma_t w = \Delta \frac{1}{\nu^2 den_{t-1} \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g^{t-1} accruals_{t-1} \right) \right]$$

reduces to

$$\Delta \left\{ \frac{F_{2t-1}}{L_{2t}} \left(cf_t - accruals_{t-1} \right) + \frac{F_{2t+1}}{L_{2t}} y_t \right\}$$

and variance of payments equals $\Delta^2 \frac{F_{2t+1}}{L_{2t}} \sigma_e^2$. ∎

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