ZERO ENTROPY OF DISTAL AND RELATED TRANSFORMATIONS

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0. INTRODUCTION

It is not too difficult, as others are aware, to prove that distal homeomorphisms of compact metric spaces which preserve a normalized Borel measure have zero entropy, if one uses Furstenberg's deep structural theorem [1]. By means of this theorem one obtains a distal minimal homeomorphism as a transfinite inverse limit of isometric extensions of the trivial homeomorphism of a single point. For each extension one verifies that the entropy is zero and by a limit theorem the proof is concluded.

The purpose of this note is to establish the theorem by a direct simple argument and to comment on a generalization which leads to a purely measure theoretic class of transformations analogous to distal homeomorphisms.

It should be noted that totally ergodic transformations with quasidiscrete spectra are representable as totally minimal affine transformations with quasidiscrete spectra, and these latter are distal [2, 3]. Moreover, the nilflows of [4] are distal. These two types of transformation are, in general, distinct [5].

1. THE MAIN THEOREM AND GENERALIZATIONS

Let (X, d) be a compact metric space and let T be a distal homeomorphism of X onto itself, that is, if $\inf_n d(T^n x, T^n y) = 0$ then $x = y$. T is said to be minimal if $TK = K$, and K closed implies $K = \phi$ or X.

THEOREM 1. If T is distal and minimal and preserves a normalized measure m, then the entropy of T with respect to m is zero ($h(T) = 0$). (The minimality condition will be dropped in Section 2.)

Proof. The case of an atomic m is dealt with

easily. We assume m is nonatomic. Let $X = S_o \supset S_1 \supset \ldots, \bigcap_n S_n = \{z\}$, S_i open, $d(S_n) \to 0$, $m(S_n) \leq r^n$ where $r < 1/e$. Let $\xi = (A_o, A_1, \ldots)$ where $A_i = S_i - S_{i-1}$, $i \geq 1$, $A_o = \{z\} \cup (S_o - S_1)$. Then ξ is a partition of X. Suppose $T^n x, T^n y \in A_{i_n}$ for some sequence $(i_1, i_2, \ldots)$ then for each N there exists n such that $T^n x \in S_N$, by the minimality of T, and therefore $T^n y \in A_{i_n} \subset S_N$ (if $N \neq 0$). In other words, $d(T^n x, T^n y)$ has infinum zero; that is, $x = y$ by the distal property. Hence $V_{i=0}^{\infty} T^{-i}\xi = \epsilon$, the partition of X into one-point sets, and

$$
\begin{aligned}
h(T) = h(T, \xi) \leq H(\xi) &= \sum_{i=0}^{\infty} -m(A_i) \log m(A_i) \\
&\leq - mA_o \log mA_o + \sum_{n=1} -nr^n \log r \qquad (1.1) \\
&\leq -\log(1 - \sum_{n=1} r^n) - \frac{r}{(1 - r)^2} \log r \\
&= \log \frac{(1 - r)}{(1 - 2r)} - \frac{r}{(1 - r)^2} \log r
\end{aligned}
$$

(Here we have used the monotonicity of $-x \log x$ on $(0, 1/e)$ and the relations $A_i \subset S_i$.) Since (1.1) is true for all $r < 1/e$ we have $h(T) = 0$. (Actually this also follows from the fact that $H(\xi) <$ and $V_{i=0} T^{-i}\xi = \epsilon$.)

Only a weak form of the distal property was used in this proof. Let us call z a separating point for T if $T^{m_n}x \to z$ and $T^{m_n}y \to z$ implies $x = y$.

The above proof yields:

THEOREM 2. If T is a homeomorphism with a separating point and if T preserves a normalized Borel measure with respect to which T is ergodic, then $h(T) = 0$.

A measure theoretic analogue of the above may be achieved as follows: Let $X = S_o \supset S_1 \supset \cdots$ be a decreasing sequence of measurable sets of positive measure such that $m(S_n) \to 0$. Such a sequence will be called a separating sieve for a measure preserving transformation T if there exists a set M of measure zero with the following property: if $x, y \in X - M$ and if for each N there exists n such that $T^n x, T^n y \in S_N$, then $x = y$; equivalently,

$$\left[\bigcap_{N=0}^{\infty} \bigcup_{n=0}^{\infty} (T \times T)^{-n}(S_N \times S_N)\right] \cap (X - M) \times (X - M) = \operatorname{diag}(X \times X) \cap (X - M) \times (X - M) \tag{1.2}$$

THEOREM 3. An ergodic transformation T of a Lebesgue space with a separating sieve has zero entropy and possesses a nonconstant eigenfunction (that is, T is not weakly mixing).

Proof. The first part is similar to the proof of Theorem 1. By (1.2) we have

$$m \times m \bigcup_{n=0}^{\infty} (T \times T)^{-n}(S_N \times S_N) \underset{N}{\to} 0$$

since $\operatorname{diag}(X \times X)$ cannot have positive measure. Consequently the $T \times T$ invariant set $\bigcup_{n=0}^{\infty} (T \times T)^{-n}(S_N \times S_N)$ is nontrivial for some N and $T \times T$ is not ergodic, that is, T is not weakly mixing.

The second part of this theorem was achieved by Furstenberg [1] for minimal distal homeomorphisms. For that case Furstenberg's method has the advantage of producing a continuous eigenfunction.

2. A DISTAL HOMEOMORPHISM HAS ZERO ENTROPY

Ellis [6] has shown that a compact metric space, on which a distal homeomorphism acts, decomposes into minimal sets; that is, $X = \cup_{\alpha \in A} X$ where $X_\alpha \cap X_\beta = \phi$ if $\alpha \neq \beta$ and X_α are closed minimal sets. However, $\{X_\alpha\}$ is not always a Hausdorff partition. Nevertheless, if $U_1, U_2, \ldots$ is a countable basis for the open sets then $V_1, V_2, \ldots$ $(V_n = \cup T^i U_n)$ form a countable basis for this partition in the sense that for $\alpha \neq \beta$ there exists n such that either

$$X_\alpha \subset V_n \quad X_\beta \subset X - V_n \quad \text{or} \quad X_\beta \subset V_n \quad X_\alpha \subset X - V_n$$

This is enough to ensure that:

(i) $\{X_\alpha\}$ is a measurable partition;

(ii) there exists a canonical system of measures m_α [7].

By virtue of [7] we have

$$h(T) = \int_{X_\zeta} h_\alpha(T_\alpha)\, dm_\zeta \qquad (2.1)$$

In (2.1) X_ζ is the factor space of X with respect to $\zeta = \{X_\alpha\}$, m_ζ is the factor measure on X_ζ, $T_\alpha = T|X_\alpha$, and h_α is the entropy of T_α with respect to m_α, which, by virtue of Theorem 1, is zero.

Consequently we have the following:

THEOREM 4. If T is a distal homeomorphism of a compact metric space X preserving a normalized Borel measure, then T has zero entropy.

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