Reprinted from JOURNAL OF MATHEMATICAL PSYCHOLOGY
All Rights Reserved by Academic Press, New York and London

Vol. 21, No. 2, April 1980
Printed in Belgium

Note

A Note on Modeling Accumulation of Information When the Rate of Accumulation Changes over Time

ROGER RATCLIFF

Dartmouth College, Hanover, New Hampshire 03755

This note presents the mathematics necessary to deal with accumulation of evidence when the rate of evidence accumulation changes during the course of processing. Accumulation of evidence is modeled by a diffusion process, the continuous analog of the random walk. The diffusion process has the advantage that a great deal of theory has been developed for continuous variables, e.g., differential equations and integration as opposed to difference equations and summation for discrete processes.

One psychological application is the recognition process (Ratcliff, 1978). In a typical paradigm, subjects study a list of items and then are tested with single probe items to which they must respond "old" if a probe was in the study list and "new" if it was not. The diffusion process represents the accumulation of evidence (goodness-of-match) that results from the comparison between a probe item and an item in memory. The drift rate of the diffusion process represents the relatedness between the probe and a memory item. To model the usual reaction time situation (information-controlled processing), two absorbing boundaries are placed $a-z$ units and z units of distance, respectively, from the starting point of the walk. If the process reaches the upper boundary (at a, if z is the starting point and 0 the lower boundary), then the process terminates with a match. If the process reaches the lower boundary, then the process terminates with a nonmatch. To model the response signal or deadline procedure (time-controlled processing), in which subjects respond at an experimenter-determined time, evidence is accumulated until the signal to respond is given. If the process is at a point above the starting point of the process when the signal is given, then a match is produced; if the process is below the starting point, then a nonmatch is produced.

The problem addressed in this note is what happens if the drift rate of the diffusion process changes. There are two cases: First, if the drift rate remains constant up to some

This research was partially supported by NSF Grant BNS 78-17471 to W. K. Estes and by a Faculty Research Grant from Dartmouth College. This paper was prepared while the author was a visiting assistant professor at Rockefeller University. I wish to thank Bill Estes for comments on this note and for the hospitality and stimulation of his laboratory. I would also like to thank Adam Reed for bringing to my attention the solution to the continuous drift case.

0022-2496/80/020178-07$02.00/0

time t_1 and then changes to some new value at t_1. Psychologically, this corresponds to the situation in which different evidence becomes available at t_1 and the rate of accumulation of evidence changes. Second, if the drift rate changes continuously as a function of time. McClelland (1979) has recently presented a model of processes in cascade. In this model, evidence can begin accumulating at stage i in the processing system before stage $i - 1$ has terminated; thus it is usual for the rate of accumulation of evidence to change as a function of time. This situation can be modeled using the diffusion process.

Time-Controlled Processing: Discrete Change in Drift

In time-controlled processing, we make the assumptions that the diffusion process is unrestricted and that the decision criterion is to respond "match" if the process is above the starting point z at evaluation time and "nonmatch" otherwise. In an unrestricted diffusion process, the probability density function $h(x, t)$ at position x at time t with drift ξ and diffusion coefficient (variance in drift) s^2 is given by:

$$h(x, t) = \frac{1}{(2\pi s^2 t)^{1/2}} \exp(-\tfrac{1}{2}(x - z - \xi t)^2 / s^2 t) \tag{1}$$

given that the process started at position z at $t = 0$ (initial condition $h(x, 0) = \delta(x - z)$). It is necessary to assume that the drift rate ξ has a distribution (i.e., there is a distribution of probe-memory item relatedness (Ratcliff, 1978)). Assuming ξ has a normal probability density function $n(u, \eta)$, the evidence, x, is distributed as

$$\begin{aligned} y(x, t) &= \int_{-\infty}^{\infty} h(x, t)\, n(u, \eta)\, d\xi \\ &= n(ut, (t(\eta^2 t + s^2))^{1/2}). \end{aligned} \tag{2}$$

Let us assume that at time t_1 relatedness changes from u_1 to u_2. There are two ways this can happen: first, comparisons in the distribution can maintain their relative drifts, for example, comparison with drift $u_1 - \frac{1}{2}\eta$ will have a new drift $u_2 - \frac{1}{2}\eta$; or second, comparisons can be randomly assigned new drift rates. For consistent drifts:

$$\begin{aligned} y(x, t) &= n(u_1 t, (t(\eta^2 t + s^2))^{1/2}) && 0 \leqslant t \leqslant t_1 \\ &= n(u_2 t + (u_1 - u_2)\, t_1, (t(\eta^2 t + s^2))^{1/2}) && t \geqslant t_1 . \end{aligned} \tag{3}$$

For nonconsistent drifts, we have to integrate over a starting value distribution of z values, $n(u_1 t_1, (t_1(\eta^2 t_1 + s^2))^{1/2})$ from (2), i.e.,

$$\begin{aligned} y(x, t) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} n(u_1 t_1, (t_1(\eta^2 t_1 + s^2))^{1/2})\, h(x, t)\, dz\right] n(u_2, \eta)\, d\xi, \qquad t \geqslant t_1 \\ &= n(u_2 t + (u_1 - u_2)\, t_1, (\eta^2 t^2 + (s^2 - 2t_1\eta^2)t + 2\eta^2 t_1^2)^{1/2}). \end{aligned} \tag{4}$$

Consider two comparison processes, one matching with relatedness $n(u_1, \eta)$, $t \leqslant t_1$ and $n(u_2, \eta)$, $t > t_1$ and one nonmatching with relatedness $n(v, \eta)$. Then, using (3), we can calculate a d' measure of discriminability:

$$d' = \frac{u_1 t - vt}{(\text{variance})^{1/2}} = \frac{d'_{\text{asy1}}}{(1 + s^2/(\eta^2 t))^{1/2}} \qquad t \leqslant t_1$$

and

$$d' = \frac{u_2 t + (u_1 - u_2)\, t_1 - vt}{(\text{variance})^{1/2}} = \frac{d'_{\text{asy2}} + (d'_{\text{asy1}} - d'_{\text{asy2}})\, t_1/t}{(1 + s^2/(\eta^2 t))^{1/2}} \qquad t \geqslant t_1 \tag{5}$$

where $d'_{\text{asy1}} = (u_1 - v)/\eta$ and $d'_{\text{asy2}} = (u_2 - v)/\eta$. Practically there is often little difference between the variance term for nonconsistent drifts [see Eq. (4)] and the variance term for consistent drifts [see Eq. (3)], but this has to be determined for the parameter values for the particular set of data to be fitted.

The formulas above allow nonmonotonic d' response signal curves to be fitted and parameters to be estimated for the time at which new information is available and for the relative asymptotic d' values for the two sets of information. The above formulas can be easily extended to case where more than two sets of relatedness are involved (with more than one discontinuity).

Information-Controlled Processing

As in the case of time-controlled processing, we assume that at time t_1, relatedness changes from u_1 to u_2. The solution hinges on finding the distribution of *nonterminated* processes at time t, $p(x, t)$. Now, $p(x, t)$ must satisfy the forward diffusion equation:

$$\frac{\partial p}{\partial t} = \frac{1}{2} s^2 \frac{\partial^2 p}{\partial x^2} - \xi \frac{\partial p}{\partial x} \tag{6}$$

subject to the boundary conditions

$$p(x, 0) = \delta(x - z) \tag{7}$$

and

$$p(a, t) = p(0, t) = 0 \qquad t > 0. \tag{8}$$

To solve (6) we use the separation of variables technique by setting $p(x, t) = X(x)\, T(t)$ (see Cox & Miller, 1965, pp. 222–223). Thus

$$X(x) \frac{\partial T(t)}{\partial t} = \frac{1}{2} s^2 T(t) \frac{\partial^2 X(x)}{\partial x^2} - \xi T(t) \frac{\partial X(x)}{\partial x};$$

therefore,

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\left[\frac{1}{2}s^2\frac{\partial^2 X(x)}{\partial x^2} - \xi\frac{\partial X(x)}{\partial x}\right] = -\lambda. \tag{9}$$

Each side of the expression in (9) must equal a constant (chosen to be $-\lambda$) because the left-hand side is a function of t alone and the right-hand side is a function of x alone and the two sides must be equal for all values of t and x. We find that using boundary condition (8),

$$p(x, t) = \sum_{n=1}^{\infty} a_n e^{kx-\lambda_n t} \sin\frac{n\pi x}{a} \tag{10}$$

is a solution of (6) with

$$k = \frac{\xi}{s^2} \quad \text{and} \quad \lambda_n = \frac{1}{2}\left(\frac{\xi^2}{s^2} + \frac{n^2\pi^2 s^2}{a^2}\right). \tag{11}$$

All that is required now is to find the coefficient a_n using boundary condition (7). Equation (10) can be rewritten as a Fourier series for $t = 0$:

$$p(x, 0)\, e^{-kx} = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right).$$

Then by the Fourier inversion formula

$$a_n = \frac{2}{a}\int_0^a e^{-kx} \sin\left(\frac{n\pi z}{a}\right) \delta(x - z)\, dx$$

(Churchill, 1963).

Thus

$$a_n = \frac{2}{a} e^{-kz} \sin\left(\frac{n\pi z}{a}\right)$$

and

$$p(x, t) = e^{\xi(x-z)/s^2} \sum_{n=1}^{\infty} \frac{2}{a} \sin\left(\frac{n\pi z}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \exp\left(-\frac{1}{2}\left(\frac{\xi^2}{s^2} + \frac{n^2\pi^2 s^2}{a^2}\right) t\right). \tag{12}$$

An alternative to (12) can be obtained using the method of images (Cox & Miller, 1967, pp. 221–222). This method is often used in the solution of problems in electromagnetic theory. For example, in finding the electric field in a system consisting of an electrical charge $+q$ at a point a with a conducting plane at position zero, the system can be replaced by a system of two electrical charges, $+q$ at a and $-q$ at $-a$. The solution to this new system is also a solution to the old system (see Reitz & Milford, 1962). By analogy, it is possible to replace the system of a diffusion process, starting point (i.e., probability source at z) and one absorbing barrier at position a with a process with a probability source at z and a probability sink at $2a - z$. For the diffusion process with two absorbing barriers, sources are placed at points $x'_n = 2na$ ($n = 0, \pm 1, \pm 2, \ldots$) with strengths $\exp(\xi x'_n/s^2)$ and sinks at points $x''_n = 2a - 2z - x'_n$ with strengths $-\exp(\xi x''_n/s^2)$. The required solution is given by:

$$p(x, t) = \frac{1}{(2s\pi^2 t)^{1/2}} \sum_{n=-\infty}^{\infty} \left[\exp\left(\frac{\xi x'_n}{s^2} - \frac{(x - z - x'_n - \xi t)^2}{2s^2 t}\right) - \exp\left(\frac{\xi x''_n}{s^2} - \frac{(x - z - x''_n - \xi t)^2}{2s^2 t}\right)\right]. \quad (13)$$

Equations (12) and (13) are alternate representations of the same function. Equation (12) converges quickly for large values of t and (13) converges quickly for small values of t.

Equations (12) and (13) were checked by comparing computed values with other explicitly known values. For example, if a becomes very large, the result approximates that of the one-boundary diffusion process and if both boundaries are moved far away from the starting point, the result approximates the unrestricted diffusion process (1). Calculated numerical values were in close agreement with exact values.

To model processes in which the rate of evidence accumulation changes at some time t_1, (12) or (13) can be used as the starting distribution for a second diffusion process with a new value of drift (see Fig. 1). For example, Ratcliff (1978, Eq. A12) provides an expression for the first passage time distribution function, $G(t, \xi)$, where ξ is drift in the diffusion process and G is a function of the starting point z (as well as other parameters). The first part of the distribution function for the overall two-component process (for $t \leqslant t_1$) can be found from $G(t, \xi)$. The second part of the distribution function (for $t > t_1$) can be found using

$$G'(t - t_1, \xi) = \int_0^a G(t - t_1, \xi)\, p(z', t_1)\, dz' \qquad t > t_1$$

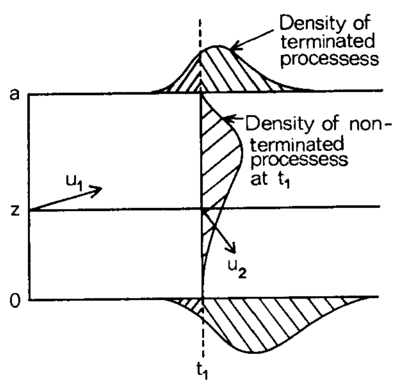

FIG. 1. An illustration of information-controlled processing in which the drift rate changes at time t_1.

where z in the expression for $G(t - t_1, \xi)$ is replaced by z'. In other words, the starting value z' in $G(t - t_1)$ is a distribution $p(z', t_1)$ and by integrating over z', the first passage time distribution $G'(t - t_1, \xi)$ can be obtained for $t > t_1$ (see Fig. 1).

Time-Controlled Processing: Continuous Change in Drift

The more general form of the diffusion equation, with drift and diffusion coefficient functions of both position and time, is called the Kolmogorov equation. In a more restricted case, if drift and the diffusion coefficient are functions of time alone, then a solution for the unrestricted diffusion process can be obtained (Gnedenko, 1967, p. 368). The Kolmogorov equation can be written as:

$$\frac{\partial p(x, t)}{\partial t} = -\xi(t) \frac{\partial p(x, t)}{\partial x} + \frac{1}{2} s^2(t) \frac{\partial^2 p(x, t)}{\partial x^2}. \tag{14}$$

If we make the change of variable:

$$x' = x - \int_0^t \xi(k)\, dk$$

and

$$t' = \int_0^t s^2(k)\, dk,$$

then (14) reduces to

$$\frac{\partial p(x', t')}{\partial t'} = \frac{1}{2} \frac{\partial^2 p(x', t')}{\partial x'^2}.$$

Solving for $p(x', t')$ with the boundary condition $p(x, 0) = \delta(x - z)$ we find

$$p(x, t) = \frac{1}{(2\pi w^2)^{1/2}} \exp\left(-\frac{1}{2} \frac{(x - z - A)^2}{w^2}\right), \tag{15}$$

where $A = \int_0^t \xi(k)\, dk$ and $w^2 = \int_0^t s^2(k)\, dk$. Equation (15) can be used instead of (1) to derive d' as a function of time as in (5) (for $t \leqslant t_1$). Reed (1973) has used a very similar model, with drift a negative exponential function of time to provide some impressive fits to forgetting functions for response signal data from a recognition memory task.

It seems that information-controlled processing (with absorbing barriers) with time variable drift parameters is intractable mathematically. Thus to model information-controlled processes we should use approximations based on the discrete case (Eqs. (12) and (13)).

Conclusion

In this note, I have addressed the problem of modeling accumulation of evidence in a diffusion model when the rate of accumulation of evidence changes during the time course of accumulation. I have provided relatively simple expressions in the case of time-controlled processing (e.g., the response signal procedure). The expressions derived for information-controlled processing are somewhat more complex and require numerical solutions. It appears that the response signal procedure (or some procedure in which accuracy as a function of processing time can be obtained) will provide the best opportunity for testing and fitting the above kind of model. In the ordinary reaction time situation, discrimination between processes will have to be done at the level of reaction time distributions and it is often likely that data will not allow fine enough discrimination. However, one results have been obtained for time-controlled processing, it may be instructive to fit the model in an ordinary reaction time situation.

The results noted here may be more generally useful than indicated above. First, the diffusion model may prove useful in modeling such paradigms as lexical priming (McClelland, 1979) or tachistoscopic paradigms in which information is assumed to be rapidly decaying from an iconic store (Rumelhart, 1970). Second, the diffusion process may be useful as a continuous approximation to discrete processes (Cox & Miller, 1965). Third, the diffusion model need not only apply to reaction time processes but may apply to more general classification tasks in which random walk classification models are appropriate.

References

Churchill, R. V. *Fourier series and boundary value problems* (2nd ed.). New York: MacGraw–Hill, 1963.

Cox, D. R., & Miller, H. D. *The theory of stochastic processes*. London: Methuen, 1965.

Gnedenko, B. V. *The theory of probability*, translated by B. D. Seckler. New York: Chelsea, 1967.

McClelland, J. L. On the time relations of mental processes: An examination of systems of processes in cascade. *Psychological Review*, 1979, **86**, 287–330.

Ratcliff, R. A theory of memory retrieval. *Psychological Review*, 1978, **85**, 59–108.

Reed, A. V. Speed-accuracy trade-off in recognition memory. *Science*, 1973, **181**, 574–576.

Reitz, J. R., & Milford, F. J. *Foundations of electromagnetic theory*. Reading, Mass.: Addison–Wesley, 1962.

Rumelhart, D. E. A multicomponent theory of the perception of briefly exposed visual displays. *Journal of Mathematical Psychology*, 1970, **7**, 191–218.

Received: August 17, 1979

Printed by the St. Catherine Press Ltd., Tempelhof 37, Bruges, Belgium