# Some algebraic properties of universal quantum semigroupoids

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CONTENTS



## I. MOTIVATION

($\Bbbk$ a field)

{ How to study symmetries? }

We have two ingredients:

1. An $A$ algebraic structure

2. the (co)actions of $H$ Group/Hopf-like Object. on $A$

| Option 1 | Option 2 |
| --- | --- |
| Fix $A$ | Study $A$ |
| $\downarrow$ | $\uparrow$ |
| Study $H$ | Fix $H$ |

* — this one — *

REMARK

We want $A$ to be "nice": $\mathbb{N}$-graded locally finite $\mathbb{k}$-algebra.

$\left( A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots \quad \& \quad \dim(A_i) < \infty \right)$

vector subspaces

"degree" preserving product

EXAMPLES of this setup:

① $A$ commutative & connected ($A_0 = \mathbb{k}$) $\xrightarrow[\text{setup}]{\text{CLASSIC}}$ $H$ group (acting by automorphisms)

(semigroup (acting by endomorphisms))

② $A$ not necessarily commutative & connected $\xrightarrow{\text{QUANTUM setup}}$ $H$ quantum group $\leftrightarrow$ Hopf algebra (co)acting.

(quantum semigroup $\leftrightarrow$ bialgebra (co)acting).

③ $A$ not necessarily commutative & not necessarily connected $\xrightarrow{\text{WEAK QUANTUM setup}}$ $H$ quantum groupoid $\leftrightarrow$ weak Hopf algebra (co)acting

(quantum semigroupoid $\leftrightarrow$ weak bialgebra (co)acting).

type ③ not ①/②

Ex. finite Quiver $\begin{cases} Q=(Q_0, Q_1, s,t: Q_1 \longrightarrow Q_0) \\ Q_0: \text{ set of vertices (finite)} \\ Q_1: \text{ set of arrows (finite)} \\ s: \text{ source map} \\ t: \text{ target map} \end{cases}$

$A = \Bbbk Q$ path algebra (more details later / Here $A_0 = \Bbbk Q_0$)

$H = \mathfrak{h}(Q)$ Hayashi's face algebra (defined later / $\mathfrak{h}(Q)$ is a weak bialgebra)

{ Our starting point }

Huang, Walton, Wicks & Won proved in arXiv:2008.00606 (2020) that:

More details in [HWWW20] or Walton's notes

$\boxed{\text{The only weak bialgebra that (co)acts universally* on } \Bbbk Q \text{ is, up to isomorphism, } \mathfrak{h}(Q).}$

*"Universal" means: $\forall$ weak bialgebra $H'$ that right coacts on $A$, $\exists!$ weak bialgebra map $\pi: H \longrightarrow H'$ such that

$$\begin{tikzcd} A \arrow[r, "act."] \arrow[dr, "act."'] & A\otimes H \arrow[d, dashed, "id\otimes\pi"] \\ & A\otimes H' \end{tikzcd}$$

They left open the following QUESTION:

> In the setup ③ above, which ring-theoretic and homological properties of $A$ are transfered to $H$ when the (co)action is universal?

**We addressed the QUESTION when $A = \Bbbk Q$.**

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# II. PRELIMINARIES

( $Q = (Q_0, Q_1, s, t)$ a finite quiver )

## 1. The path algebra

$\Bbbk Q$ Path algebra

$\Bbbk$-algebra generated by $\{e_i\}_{i \in Q_0}$ and $\{p\}_{p \in Q_1}$

PRODUCT: $m(e_i \otimes e_j) = \delta_{ij}\, e_i \,, \quad i,j \in Q_0$

$m(p \otimes q) = \delta_{t(p),s(q)}\, pq \,, \quad p,q \in Q_1$

UNIT: $1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$

we read paths from left to right.

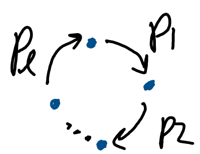
$\Bbbk Q$ is $\mathbb{N}$-graded by path length, that is,

$$(\Bbbk Q)_\ell = \Bbbk Q_\ell \quad , \quad Q_\ell : \text{paths of length } \ell.$$

In particular, $(\Bbbk Q)_0 = \Bbbk Q_0$. Thus $\Bbbk Q$ is non-connected when $|Q_0| > 1$.

## 2. Concepts on quivers.

- A path $p_1 p_2 \cdots p_\ell$ is called a cycle if $s(p_1) = t(p_\ell)$.



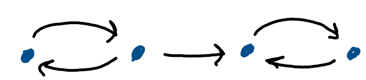

- $Q$ is said to be acyclic if it has no cycles.

- A cycle is said to be source/sink if there is an arrow leaving/entering the cycle.

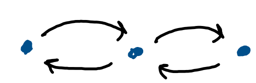

- A cycle is said to be exclusive if it is disjoint from every other cycle.

- $Q$ is said to satisfy the exclusive condition if every cycle of $Q$ is exclusive.

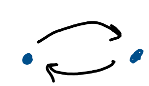
Ex.



non-isolated cycles
exclusive cycles



non-isolated cycles
non-exclusive cycles



isolated cycle
exclusive cycle.

- A cycle is called isolated if it is neither source nor sink.

- Q is said to be connected if the underlying graph is connected.
- Q is said to be strongly connected if any two distinct vertices are connected by a path.

Ex.: $\bullet \to \bullet$ connected
not strongly connected.

NOTICE: Isolated $\Rightarrow$ exclusive.

NOTICE: strongly connected $\Rightarrow$ connected.

- Q is said to be path reversible if any path has $\underbrace{\text{reverse}}_{\text{not necessarily of the same length}}$.

# 3 Kronecker square of a quiver

$\hat{Q}$ Kronecker square

Is the quiver $\hat{Q} = (\hat{Q}_0, \hat{Q}_1, \hat{s}, \hat{t})$ given by

$$\hat{Q}_0 = \{[i,j]\}_{i,j\in Q_0} \quad , \quad \hat{Q}_1 = \{[p,q]\}_{p,q\in Q_1}$$

$$\hat{s}([p,q]) = [s(p), s(q)] \qquad \hat{t}([p,q]) = [t(p), t(q)]$$

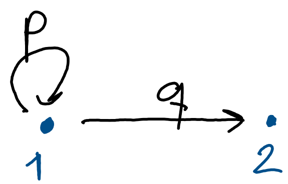
NOTICE:

$|\hat{Q}_0| = |Q_0|^2.$

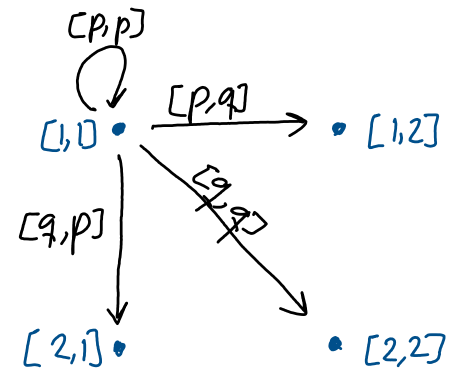
$|\hat{Q}_1| = |Q_1|^2$

$Q \xhookrightarrow[\text{subquiver}]{} \hat{Q}.$

Ex.




$Q$




$\hat{Q}$

# 4. Hayashi's face algebra

$h(Q)$
Hayashi's face algebra
[Hayashi, 96]

$h(Q) = \Bbbk\langle x_{i,j}\, ,\, x_{p,q} \mid i,j \in Q_0\, ,\, p,q \in Q_1 \rangle / (R)$ with relations $R$ given by

$$x_{p,q}\, x_{p',q'} = \delta_{t(p),s(p')}\, \delta_{t(q),s(q')}\, x_{p,q}\, x_{p',q'}$$

$$x_{s(p),s(q)}\, x_{p,q} = x_{p,q} = x_{p,q}\, x_{t(p),t(q)}$$

$$x_{i,j}\, x_{k,\ell} = \delta_{i,k}\, \delta_{j,\ell}\, x_{i,j}$$

for all $p,p',q,q' \in Q_1$ and $i,j,k,\ell \in Q_0$.

$h(Q)$ is a $\Bbbk$-algebra with unit $1_{h(Q)} = \sum_{i,j \in Q_0} x_{i,j}$.

(In fact, as mentioned before, $h(Q)$ is a weak bialgebra).

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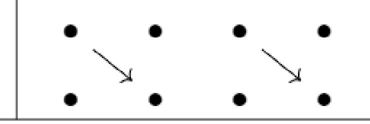
# III. RESULTS

MAIN THEOREM [C.-Walton, 2020]

As unital $\Bbbk$-algebras, $h(Q) \cong \Bbbk\hat{Q}$.

Consequence 1: Calculation of $\mathfrak{h}(Q)$ for particular cases:

| $Q$ | $\Bbbk Q$ | $\widehat{Q}$ | $\Bbbk\widehat{Q} \simeq \mathfrak{H}(Q)$ |
|---|---|---|---|
| $\bullet_1 \quad \bullet_2 \quad \cdots \quad \bullet_n$ | $\Bbbk^n$ | $\begin{matrix} \bullet_{[1,1]} & \cdots & \bullet_{[1,n]} \\ \vdots & \ddots & \vdots \\ \bullet_{[n,1]} & \cdots & \bullet_{[n,n]} \end{matrix}$ | $\Bbbk^{n^2}$ |
| $\begin{tikzcd} \bullet \arrow[loop above, "p_2"] \arrow[loop right, "p_1"] \arrow[loop left, "\cdots"] \arrow[loop below, "p_n"] \end{tikzcd}$ | $\Bbbk\langle t_1, \dots, t_n\rangle$ | $\begin{tikzcd} \bullet \arrow[loop above, "\cdots"] \arrow[loop left, "{[p_i,p_j]}"] \arrow[loop right, "{[p_1,p_1]}"] \arrow[loop below, "{[p_n,p_n]}"] \end{tikzcd}$ | $\Bbbk\langle t_{i,j}\rangle_{i,j=1}^{n}$ |
| $\bullet \longrightarrow \bullet$ | $T_2(\Bbbk) = \begin{bmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{bmatrix}$ | $\begin{tikzcd} \bullet \arrow[dr] & \bullet \\ \bullet & \bullet \end{tikzcd}$ | $T_2(\Bbbk) \times \Bbbk^2$ |
| $\bullet \longrightarrow \bullet \longrightarrow \bullet$ | $T_3(\Bbbk) = \begin{bmatrix} \Bbbk & \Bbbk & \Bbbk \\ 0 & \Bbbk & \Bbbk \\ 0 & 0 & \Bbbk \end{bmatrix}$ | $\begin{tikzcd} \bullet \arrow[dr] & \bullet \arrow[dr] & \bullet \\ \bullet \arrow[dr] & \bullet \arrow[dr] & \bullet \\ \bullet & \bullet & \bullet \end{tikzcd}$ | $T_3(\Bbbk) \times T_2(\Bbbk)^2 \times \Bbbk^2$ |
| $\bullet \longrightarrow \bullet \longleftarrow \bullet$ | $\begin{bmatrix} \Bbbk & 0 & 0 \\ \Bbbk & \Bbbk & 0 \\ \Bbbk & 0 & \Bbbk \end{bmatrix}$ | $\begin{tikzcd} \bullet \arrow[dr] & \bullet & \bullet \arrow[dl] \\ \bullet & \bullet & \bullet \\ \bullet \arrow[ur] & \bullet & \bullet \arrow[ul] \end{tikzcd}$ | $\begin{bmatrix} \Bbbk & 0 & 0 & 0 & 0 \\ \Bbbk & \Bbbk & 0 & 0 & 0 \\ \Bbbk & 0 & \Bbbk & 0 & 0 \\ \Bbbk & 0 & 0 & \Bbbk & 0 \\ \Bbbk & 0 & 0 & 0 & \Bbbk \end{bmatrix} \times \Bbbk^4$ |
| $\bullet \rightrightarrows \bullet$ | $\begin{bmatrix} \Bbbk & \Bbbk^2 \\ 0 & \Bbbk \end{bmatrix}$ | $\begin{tikzcd} \bullet \arrow[dr, shift left=3] \arrow[dr, shift left=1] \arrow[dr, shift right=1] \arrow[dr, shift right=3] & \bullet \\ \bullet & \bullet \end{tikzcd}$ | $\begin{bmatrix} \Bbbk & \Bbbk^4 \\ 0 & \Bbbk \end{bmatrix} \times \Bbbk^2$ |
| $\begin{tikzcd} \bullet \arrow[loop above] \arrow[r] & \bullet \end{tikzcd}$ | $\begin{bmatrix} \Bbbk[t] & \Bbbk[t] \\ 0 & \Bbbk \end{bmatrix}$ | $\begin{tikzcd} \bullet \arrow[loop above] \arrow[r] \arrow[d] \arrow[dr] & \bullet \\ \bullet & \bullet \end{tikzcd}$ | $\begin{bmatrix} \Bbbk[t] & \Bbbk[t] & \Bbbk[t] & \Bbbk[t] \\ 0 & \Bbbk & 0 & 0 \\ 0 & 0 & \Bbbk & 0 \\ 0 & 0 & 0 & \Bbbk \end{bmatrix} \times \Bbbk^2$ |
| $\bullet \rightarrow \bullet \qquad \bullet \rightarrow \bullet$ | $T_2(\Bbbk)^2$ | $\begin{tikzcd} \bullet \arrow[dr] & \bullet & \bullet \arrow[dr] & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{tikzcd}$ | $T_2(\Bbbk)^4$ |



**Consequence 2:** Transference of ring-theoretic and homological properties:

GAME

$$
\boxed{
\begin{array}{ccc}
\text{Alg-property on } \Bbbk Q & \Longleftrightarrow & \text{Quiv-property on } Q \\
 & & \Updownarrow \\
\text{Alg-property on } h(Q) \cong \Bbbk \widehat{Q} & \Longleftrightarrow & \text{Quiv-property on } \widehat{Q}
\end{array}
}
$$

$$
\boxed{
\begin{array}{ccc}
\Bbbk Q \text{ is finite dimensional} & \overset{\text{classic}^*}{\Longleftrightarrow} & Q \text{ is acyclic} \\
\Updownarrow & & \Updownarrow \text{ new} \\
h(Q) \cong \Bbbk \widehat{Q} \text{ is finite dimensional} & \underset{\text{main thm.}}{\Longleftrightarrow} & \widehat{Q} \text{ is acyclic}
\end{array}
}
$$

*eg. [Assem et. al, 06]

Let $A$ be a finitely generated $\Bbbk$-algebra. The GKdim of $A$

is defined by $\mathrm{GKdim}(A) = \sup_{V} \varlimsup_{n\to\infty} \log_n(\dim_{\mathbb{K}} V^n)$    $V$: finite dim. subspace
$V^n = \langle v_1 \cdots v_n \mid v_i \in V \rangle$

$\mathbb{K}Q$ has finite GKdim $\overset{*}{\Longleftrightarrow}$ $Q$ satisfies the exclusive condition

$\Updownarrow$    $\Updownarrow$ new

$h(Q) \equiv \mathbb{K}\hat{Q}$ has finite GKdim $\underset{\text{main thm.}}{\Longleftrightarrow}$ $\hat{Q}$ satisfies the exclusive condition

In this case, $\mathrm{GKdim}(h(Q)) = \mathrm{GKdim}(\mathbb{K}Q) =$ maximal lenght of chains of cycles in $Q$.

* [Moreno, Siles, 08]

Let $R$ be a ring. A right $R$-module $M$ is said to be Noetherian if every submodule of $M$ is finitely generated. In particular, $R$ is said to be right Noetherian if $R_R$ is Noetherian. Likewise, we can define left Noetherian rings.

$\Bbbk Q$ is left/right Noetherian $\overset{\text{classic}*}{\iff}$ $Q$ has no source/sink cycles.

$\Updownarrow$ $\Updownarrow$ new

$h(Q) \cong \Bbbk\hat{Q}$ is left/right Noetherian $\underset{\text{main thm.}}{\iff}$ $\hat{Q}$ has no source/sink cycles

* e.g. [Ufnarovski, 82]

A ring $R$ is called semiprime if it has no nonzero nilpotent right ideals ($I$ ideal, $I^n = 0 \Rightarrow I = 0$)

$\Bbbk Q$ is semiprime $\overset{*}{\iff}$ $Q$ is path reversible

$\Updownarrow$ $\Updownarrow$ new

$h(Q) \cong \Bbbk\hat{Q}$ is semiprime $\underset{\text{main}}{\iff}$ $\hat{Q}$ is path reversible thm.

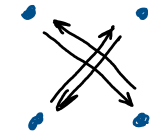
*[Siles, 08]

A ring $R$ is called prime if all nonzero right ideals $I, J$ of $R$ satisfy $IJ \neq 0$.

$$\begin{tikzcd}
\text{$\Bbbk Q$ is prime \& $Q$ has a loop} \arrow[r, Leftrightarrow, "\text{classic}^*"] \arrow[d, Rightarrow] & \text{$Q$ is strongly connected \& $Q$ has a loop.} \arrow[d, Rightarrow, "\text{new}"] \\
\text{$h(Q) \cong \Bbbk\widehat{Q}$ is prime} \arrow[r, Leftrightarrow, "\text{main thm.}"'] & \text{$\widehat{Q}$ is strongly connected.}
\end{tikzcd}$$

*e.g. [Crawley-Boevey, 92]

NOTICE: $Q$ strongly connected $\not\Rightarrow$ $\hat{Q}$ strongly connected. For example

$Q$:  strongly connected $\qquad$ $\hat{Q}$:  not strongly connected.

$$\boxed{\begin{array}{ccc} h(Q) \cong \Bbbk\hat{Q} \text{ is prime} & \overset{\text{classic}}{\Longleftrightarrow} & \hat{Q} \text{ strongly connected} \\ \Downarrow & & \Downarrow \text{ new.} \\ \Bbbk Q \text{ is prime.} & \underset{\text{main thm.}}{\Longleftrightarrow} & Q \text{ strongly connected} \end{array}}$$

The *projective dimension* of a module $M_R$, written $\mathrm{pd}(M_R)$, is the shortest length $n$ of a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M_R \to 0$$

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$

or $\infty$ if not finite projective resolution exists. The *right global dimension* of $R$ is defined by

$$\mathrm{rgld}(R) := \sup\{\mathrm{pd}(M) \mid M \text{ any right } R\text{-module}\}.$$

Likewise, $\mathrm{lgld}(R)$ is defined; when $\mathrm{rgld}(R) = \mathrm{lgld}(R)$ (e.g., when $R = A$ is a finite-dimensional $\Bbbk$-algebra) we simply write $\mathrm{gld}(R)$. A $\Bbbk$-algebra $A$ is called *hereditary* if $\mathrm{rgld}(A) \leq 1$ (equivalently, if $\mathrm{lgld}(A) \leq 1$).

$$\boxed{\mathrm{gldim}(\Bbbk Q) = \mathrm{gldim}(h(Q))}$$

- Any path algebra has $\mathrm{gldim} \leq 1$. classic e.g. [Assem et. al., 06]
- $\mathrm{gldim} = 0 \iff Q$ is arrowless
- $Q$ is arrowless $\overset{\text{new}}{\iff}$ $\hat{Q}$ is arrowless

Finally, a $\mathbb{N}$-graded $\Bbbk$-algebra $A$ is said to be *Koszul* if it has a linear minimal graded free resolution, that is, there exists an exact sequence

$$\cdots \to A(-i)^{b_i} \to \cdots \to A(-2)^{b_2} \to A(-1)^{b_1} \to A \to \Bbbk \to 0,$$

where $A(-j)$ is the graded algebra $A$ with grading shifted up by $j$, $A(-j)_i = A_{i-j}$, and the exponents $b_i$ refer to the $b_i$-fold direct sum.

$\boxed{\Bbbk Q \text{ and } \hbar(Q) \text{ are Koszul}}$

- Any tensor algebra is Koszul classic.
- $\Bbbk Q \cong T_{\Bbbk Q_0}(\Bbbk Q_1)$, $\hbar(Q) \underset{\text{main thm.}}{\cong} \Bbbk \widehat{Q} \cong T_{\Bbbk \widehat{Q}_0}(\Bbbk \widehat{Q}_1)$

Notation Denote by

- $C = (c_{ij})_{i,j \in Q_0}$ the adjacency matrix of (arrows in) $Q$
- $C^k = (c_{ij}^{(k)})_{i,j \in Q_0}$ the adjacency matrix of paths of length $k$ in $Q$.

$$\dim_{\Bbbk} \Bbbk Q \overset{\text{classic}}{=} \sum_{\substack{i,j \in Q_0 \\ k \geq 0}} c_{ij}^{(k)}, \quad \dim_{\Bbbk} h(Q) \overset{\text{new}}{=} \sum_{\substack{i,j \in Q_0 \\ k \geq 0}} \left(c_{ij}^{(k)}\right)^2$$

Let $I$ be a finite set and $R := \bigoplus_{i \in I} \Bbbk$ be the algebra of $\Bbbk$-valued functions on $I$. Recall an $R$-bimodule $M$ is called $\mathbb{N}$-*graded* if it has a $\Bbbk$-vector space decomposition $M = \bigoplus_{n \in \mathbb{N}} M_n$ such that $M_n \cdot M_m \subseteq M_{n+m}$. We say that $M = \bigoplus_{n \in \mathbb{N}} M_n$ is locally finite if each $M_n$ is finite-dimensional. When $R = \Bbbk$, this is the usual definition of a $\mathbb{N}$-graded $\Bbbk$-algebra. Also, note that any $R$-bimodule $N$ can be seen as an $I \times I$-graded vector space $N = \bigoplus_{i,j \in I} N_{i,j}$.

We define the *(matrix) Hilbert series $h_M(t)$ of $M$* to be a matrix-valued series with entries given by

$$h_M(t)_{i,j} = \sum_{k=0}^{\infty} \dim((M_k)_{i,j}) t^k.$$

$$H_{\Bbbk Q}(t) \overset{\text{classic}}{=} (I - Ct)^{-1} = I + Ct + C^2t^2 + C^3t^3 + \cdots$$

new

$H_{h(\otimes)}(t) = (I \otimes I) + (C \otimes C)t + (C^2 \otimes C^2)t^2 + \cdots$

| $Q$ | $\dim_{\Bbbk} \Bbbk Q$ | $\dim_{\Bbbk} \mathfrak{H}(Q)$ |
|---|---|---|
| $\bullet_1 \longrightarrow \bullet_2 \longrightarrow \cdots \longrightarrow \bullet_n \quad n \geq 1$ | $\dfrac{n(n+1)}{2}$ | $\dfrac{n(n+1)(2n+1)}{6}$ |
| $\bullet_1 \longrightarrow \bullet_2 \longleftarrow \cdots \longleftarrow \bullet \longrightarrow \bullet_{2n}$ or $\bullet_1 \longrightarrow \bullet_2 \longleftarrow \cdots \longrightarrow \bullet \longleftarrow \bullet_{2n-1}$ $\quad n \geq 2$ | $2n-1$ | $2n^2-2n+1$ |
| $\begin{array}{lll} \bullet_1 & & \\ & \searrow & \\ & & \bullet_3 \longrightarrow \cdots \longrightarrow \bullet_n \\ & \nearrow & \\ \bullet_2 & & \end{array} \quad n \geq 4$ | $\displaystyle\sum_{k=2}^{n} k$ | $\displaystyle\sum_{k=2}^{n} k^2$ |
| $\begin{array}{lll} \bullet_1 & & \\ & \nwarrow & \\ & & \bullet_3 \longrightarrow \cdots \longrightarrow \bullet_n \\ & \swarrow & \\ \bullet_2 & & \end{array} \quad n \geq 4$ | $2n-1+\displaystyle\sum_{k=1}^{n-4} k$ | $n^2+(n-1)^2+\displaystyle\sum_{k=1}^{n-4} k^2$ |
| $\begin{array}{ccccc} & & \bullet & & \\ & & \uparrow & & \\ \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \end{array}$ | 19 | 87 |
| $\begin{array}{cccccc} & & \bullet & & & \\ & & \uparrow & & & \\ \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \end{array}$ | 25 | 131 |
| $\begin{array}{ccccccc} & & \bullet & & & & \\ & & \uparrow & & & & \\ \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \longrightarrow & \bullet \end{array}$ | 32 | 188 |

# IV FUTURE WORK

HWWW(2020) also contains a characterization of weak bialgebras acting universally on quotients of path algebra.

{Our next step} $\boxed{\text{How behaves this transference of properties with quotients of path algebras? In particular, with preprojective algebras.}}$

Stay tuned for progress on this!

Thank you

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