DERIVED FUSION CATEGORIES

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The use of tilting modules has shown its power to endow certain semisimple categories with a "reduced" tensor product structure.

This categories produce invariants of 3-manifolds.

This construction is given by Andersen and Paradowski, it is based in the concept of tilting modules. But, can we avoid the use of tilting modules?

This is the question we want to address

PRELIMINARIES:

Different versions of sl_2 over $\mathbb{C}$:

Classical:

sl_2: $\mathbb{C}$-vector space spanned by E, F, H plus relations
$[E,F]=H$; $[H,E]=2E$; $[H,F]=-2F$

$U(sl_2)$: Associative $\mathbb{C}$-algebra with unit generated by E, F, H + $EF-FE=H$; $HE-EH=2H$; $HF-FH=-2F$

Quantum:

$U_v(sl_2)$: v indeterminate. Associative $\mathbb{C}(v)$-algebra with unit generated by E, F, K, K^{-1} +

generic version

$KK^{-1}=K^{-1}K=1$; $EF-FE=\dfrac{K-K^{-1}}{v-v^{-1}}$;
$KEK^{-1}=v^2E$; $KFK^{-1}=v^{-2}F$

$U_q(sl_2)$: $q\in\mathbb{C}\setminus\{0,\pm1\}$

Lusztig version

$\mathcal{A}=\mathbb{Z}[v,v^{-1}]$; $U_\mathcal{A}$ $\mathcal{A}$-subalgebra of $U_v(sl_2)$ generated by $E^{(i)}=\dfrac{E^i}{[i]!}$; $F^{(j)}=\dfrac{F^j}{[j]!}$; $[n]=\dfrac{v^n-v^{-n}}{v-v^{-1}}$

Consider $\mathbb{C}$ as an $\mathcal{A}$-algebra $v\mapsto q$; $\boxed{U_q := U_\mathcal{A}\otimes_\mathcal{A}\mathbb{C}}$

REPRESENTATION THEORY

$\mathcal{U}$ — category of f.d. U_q-modules $(q \in \mathbb{C} \setminus \{0, \pm 1\})$

Thm: $\mathcal{U}$ is semisimple category iff q is not a root of unity.

- If q not root of unity $\Rightarrow \mathcal{U} \equiv$ f.d. $\mathfrak{g}$-mod ($\mathfrak{sl}_2$-mod)
- when q root of unity, we get a more interesting category. Study this case!

The interesting case is when q is a root of unity.

For the rest of the talk, let q be a primitive ℓ^{th}-root of unity.

In $\mathcal{U}$ any module is the sum of its weight spaces

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

Highest weight modules:

$\Delta_q(n)$: Standard modules $\longrightarrow$ quantizations of simple rep's

$\nabla_q(n)$: Costandar modules $\longrightarrow$ $\nabla_q(n) = \Delta_q(n)^*$

$L_q(n)$: Simple modules : $\Delta_q(n) \twoheadrightarrow L_q(n)$

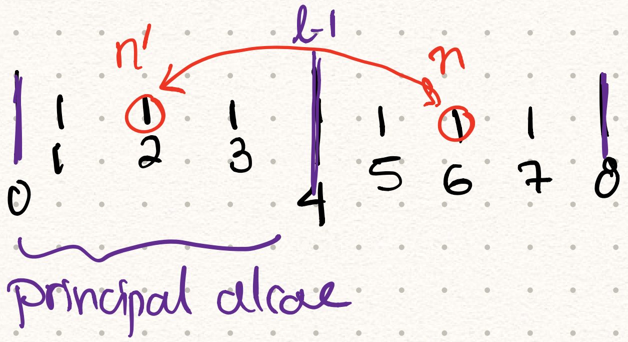
For $\boxed{0 \leq n \leq l-2}$; $\Delta_q(n) = \nabla_q(n) = L_q(n)$

If $n \equiv (-1) \bmod (l)$; $\Delta_q(n) = L_q(n)$

Exact sequence for standard modules:

$$0 \to L_q(n') \longrightarrow \Delta_q(n) \longrightarrow L_q(n) \longrightarrow 0$$

q is 5-root of unity.



All of the previous are the indecomposable.

$P_q(n)$; $I_q(n)$:

$\underbrace{P_q(n) \; ; \; I_q(n)}$ projectives and injectives. Proj = Inj.

$$P_q(n) = I_q(n)^*$$

Tilting modules:

Let M be a finite dimensional U_q-module.

M has a standard (costandard) filtration if there exists chain of submodules $0 = V_0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = M$ such that $V_r / V_{r-1} \cong \Delta_q(n_r)$ $\left(V_r / V_{r-1} \cong \nabla_q(n_r) \right)$

$T \in \mathcal{U}$ is tilting if it has a std and cstd filtration.

Properties of tilting modules:

Let $\mathcal{T}$ be the full subcategory of $\mathcal{U}$ consisting of tilting modules.

- $\mathcal{T}$ Krull-Schmidt category.
- Closed under $\otimes$, $\oplus$, duals.
- For $\mathfrak{sl}_2$: $0 \to \Delta_q(n) \to \boxed{T_q(n)} \to \Delta_q(n') \to 0$
- For any n, there exists indecomposable tilting module $T_q(n)$

Because the category $\mathcal{T}$ is Krull-Schmidt, for any $T \in \mathcal{T}$ we have

$$T \cong \underbrace{\boxed{\bigoplus_{n=0}^{l-2} \underbrace{T_q(n)}_{\text{simple}}^{\hat{n}}}}_{\text{semisimple part}} \oplus \cancel{\underbrace{\bigoplus_{n \geq l-1} \boxed{T_q(n)^{\hat{n}}}}_{\text{non-semisimple part}}}$$

has quantum dim = 0

Fusion part of $\mathcal{T}$

This modules are called negligible modules.

Negligible modules:

A finite dimensional U_q-module M is called negligible if:

For any $f \in End_U(M)$ we have

For $T_q(n)$: it is negligible iff $n \geq l-1$

$$\underbrace{Tr_q}(f) = 0$$

$$:= \underset{\text{left trace}}{Tr(Kf)} = \underset{\text{right trace}}{Tr(K^{-1}f)} \qquad \left(= Tr\Big(\prod_{\beta \in \Phi^+} K_\beta f\Big)\right)$$

When M is indecomposable, M negligible iff $\overbrace{\dim_q(M)}^{Tr_q(1_M)} = 0$

A semisimple category of tilting modules:

Consider the irreducible tilting modules $L_q(n)$, $0 \le n \le l-2$

(Recall that in this situation $L_q(n) \cong \Delta_q(n) = T_q(n)$)

$\mathcal{F}$ full subcat of $\mathcal{T}$ whose objects are direct sum of $L_q(n)$; $0 \le n \le l-2$.

$\mathcal{F}$ is semisimple abelian category.

But: Not closed for $\otimes$

$F_1, F_2 \in \mathcal{F}$

$$F_1 \otimes F_2 = \bigoplus_{n=0}^{l-2} L_q(n) \oplus \cancel{\bigoplus_{n \ge l-1} L_q(n)}$$

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define a reduce $\otimes$-product

$$\boxed{F_1 \overline{\otimes} F_2 = \bigoplus_{n=0}^{l-2} L_q(n)}$$

$\mathcal{F}$:
- ss, abelian
- closed $\oplus$, $\overline{\otimes}$; dual.
- has finitely many iso-classes of simples.
- $\overline{\otimes}$ commutative and associative.

$\}$ $\mathcal{F}$ is a Fusion category.

What do we need to generalize this idea?

Let $\mathcal{C}$ be a category such that:

- rigid monoidal tensor cat.
- with $\mathbb{1}$; $\mathbb{C} = \mathrm{End}_{\mathcal{C}}(\mathbb{1})$
- Pivotal $X \cong X^{**}$
- Traces Tr_L; Tr_R

$$Tr_L = Tr_R = Tr_q$$

$\boxed{\text{Spherical cat}}$

In spherical categories we have the notion of negligible objects:

$$X \text{ negligible iff } Tr_q(f) = 0 \quad \forall f \in \mathrm{End}(X)$$

But... we don't have tilting modules

Let $\mathcal{N}_{\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ consisting of negligible modules.

- $\mathcal{C}/\mathcal{N}_{\mathcal{C}}$ $\to$ has infinitely many simples
- $\mathcal{T}/\mathcal{N}_{\mathcal{T}}$ $\to$ Fusion cat. Tilting., $\boxed{\text{reduced } \otimes.}$
  finitely many simples. $\quad$ $\hookrightarrow$ negligible modules
- $\mathcal{U}/\mathcal{N}_{\mathcal{U}}$ $\to$ infinitely many simples.

DERIVED APPROACH:

$D^b(\mathcal{U})$: Derived cat. of $\mathcal{U}$

$K^b(\mathcal{T})$: Homotopy cat of $\mathcal{T}$

Theorem (BBM): There is a triangle equivalence between

$$D^b(\mathcal{U}) \equiv K^b(\mathcal{T})$$

We fixed the following notation:

- $\mathrm{Trg}(\mathcal{A})$: Triangulated subcategory of $D^b(\mathcal{U})$ generated by $\mathcal{A}$.
- $\langle \mathcal{A} \rangle$: Smallest full triangulated subcategory of $D^b(\mathcal{U})$ that contains $\mathcal{A}$ and is closed under retracts and tensor product with arbitrary modules.

Some properties:

•) $K^b(\mathcal{W}_T)$ is closed under retracts.

•) If we identify $K^b(\mathcal{W}_T)$ with $\mathrm{Trg}(\mathcal{W}_T)$ under BBM we have $\mathrm{Trg}(\mathcal{W}_T) = \langle \mathcal{W}_T \rangle$

•) $K^b(T) \equiv D^b(\mathcal{U})$: $T/\mathcal{W}$

$$K^b(T)/K^b(\mathcal{W}_T) \equiv D^b(\mathcal{U})/\langle \mathcal{W}_T \rangle$$

tiltings

•) $K_0\left(D^b(\mathcal{U})/\langle \mathcal{W}_T \rangle\right) \equiv K_0(T/\mathcal{W}) = K_0(\mathcal{F})$ Fusion ring.

Avoiding tilting modules:

We have the following characterization for the tensor ideal $\langle \mathcal{N}_T \rangle$

$\langle \mathcal{N}_T \rangle \equiv$ smallest full Δ-subcat $D^b(\mathcal{U})$ generated by M in a $\boxed{\text{singular block}}$. Closed under retracts and is a tensor ideal. $\langle D^b_{sing} \rangle$

Here tiltings

no tilting.

Then, the Verdier quotient

$$K_0\left(D^b(\mathcal{U}) \big/ \langle D^b_{sing} \rangle \right)$$

Fusion ring.

Moreover, we have $K_0\left(D^b(\mathcal{U}) / \langle D^b_{sing} \rangle \right) \cong R$

Fusion ring.

Can we avoid the use of the intrinsic representation theory of $\mathcal{U}_q$?

What if we consider $\mathcal{N}_{\mathcal{U}}$ all negligible modules for $\mathcal{U}$?

$$D^b(\mathcal{U}) \big/ \langle \mathcal{N}_{\mathcal{U}} \rangle$$

Using all the negligible modules:

We can consider the category $D^b(\mathcal{U})/\langle \mathcal{N}_{\mathcal{U}} \rangle$

But saddly I do not know how big $\langle \mathcal{N}_{\mathcal{U}} \rangle$ is.

- If $\boxed{Trg(\mathcal{N}_{\mathcal{U}}) \text{ is closed under retracts}} \Rightarrow Trg(\mathcal{N}_{\mathcal{U}}) = \langle \mathcal{N}_{\mathcal{U}} \rangle$ it is a proper ideal and $K_0\left(D^b(\mathcal{U})/\langle \mathcal{N}_{\mathcal{U}} \rangle\right)$ = Fusion ring.

- If I consider small quantum group for sl_2, the above works.

The above, suggest the study of the following:

If $\mathcal{C}$ abelian spherical category

$D^b(\mathcal{C})/\langle \mathcal{N}_{\mathcal{C}} \rangle \longrightarrow$ Derived fusion category

$K_0\left(D^b(\mathcal{C})/\langle \mathcal{N}_{\mathcal{C}} \rangle\right) \to$ Derived fusion ring.

And that's all

THANK YOU FOR THE ATTENTION!