

# Realization of multi-tensor $C^*$ -categories :

and (semisimple, rigid)  $C^*$ -bicategories

## Quantum Symmetries Student Seminar

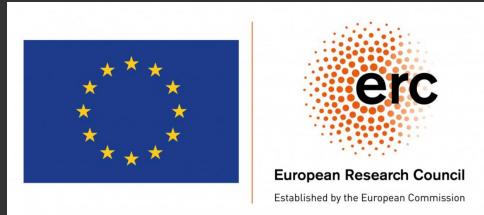
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background : [Yam04], [BKLR15], [GL19]



## Multi-tensor $C^*$ -categories

$\mathcal{C}$  (small) category with objects  $X, Y, Z, \dots$  arrows  $r, s, t, \dots$

$$r: X \rightarrow Y$$

$$s \circ r: X \rightarrow Z$$

$$\text{s.t. } t \circ (s \circ r) = (t \circ s) \circ r$$

associative composition

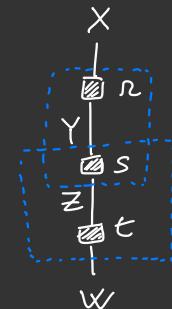
$$1_Y: Y \rightarrow Y$$

$$\text{s.t. } 1_Y \circ r = r, \quad s \circ 1_Y = s$$

identity arrow

we shall denote  $1_X = \begin{array}{c} X \\ \square \\ X \end{array}$  just by  $\begin{array}{c} X \\ | \\ X \end{array}$

and morally identify  $X$  with  $1_X$  ("only arrows approach")

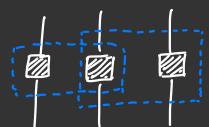


$$\begin{array}{c} X \\ \square \\ Y \\ \square \\ Y \end{array} = \begin{array}{c} X \\ \square \\ Y \end{array}$$

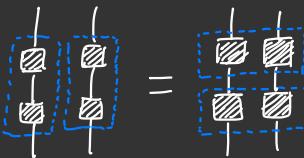
tensor structure :  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  bifunctor  
 (strict)

$X \otimes Y$ ,  $r \otimes t$

$$\begin{array}{c} X \quad Y \\ | \quad | \\ r \quad t \\ z \quad w \end{array}$$



strictly associative  $\otimes$



functionality of  $\otimes$

$$1_X \otimes 1_Y = 1_{X \otimes Y}$$

in particular:

$$\begin{array}{c} | \quad | \\ \square \quad \square \end{array} = \begin{array}{c} | \quad | \\ \square \quad | \end{array} = \begin{array}{c} | \quad | \\ | \quad \square \end{array}$$

"slide up and down"

$$\begin{array}{c} I \\ | \quad | \\ \square \quad a \\ | \quad b \end{array} = \begin{array}{c} I \quad I \\ | \quad | \\ \square \quad \square \end{array} = \begin{array}{c} I \quad I \\ | \quad | \\ | \quad \square \end{array} = \begin{array}{c} I \\ | \quad | \\ \square \quad b \\ | \quad a \end{array}$$

$\text{Hom}_{\mathcal{C}}(I, I)$  is abelian under  $\circ$

Aside: by adding one more layer down:  $N, M, L, \dots$  regions or 0-objects

$X, Y, Z, \dots$  objects  $\rightsquigarrow$  1-arrows

$r, s, t, \dots$  arrows  $\rightsquigarrow$  2-arrows

$$N = \boxed{\phantom{X}}, \quad M = \boxed{\begin{array}{|c|} \hline \text{X} \\ \hline \end{array}}, \quad L = \boxed{\begin{array}{|c|} \hline \text{Y} \\ \hline \end{array}}$$

gives a (strict) 2-category

$$M \boxed{\begin{array}{|c|} \hline \text{X} \\ \hline \end{array}} \quad N, \quad N \boxed{\begin{array}{|c|} \hline \text{Y} \\ \hline \end{array}} \quad L$$

then

$$M \boxed{\begin{array}{|c|} \hline \text{X} \otimes \text{Y} \\ \hline \end{array}} \quad L$$

partially defined

$$M \xleftarrow{\text{X}} N \quad N \xleftarrow{\text{Y}} L$$

$$M \xleftarrow{\text{X} \otimes \text{Y}} L$$

$$M \cdot \xleftarrow{\text{z}} \text{r} \cdot \xleftarrow{\text{w}} \text{t} \cdot L$$

$$M \boxed{\begin{array}{|c|} \hline \text{X} \\ \hline \end{array}} \quad N \boxed{\begin{array}{|c|} \hline \text{Y} \\ \hline \end{array}} \quad L \boxed{\begin{array}{|c|} \hline \text{Z} \\ \hline \end{array}}$$

$= s \circ r : X \Rightarrow Z : N \rightarrow M$

vertical

$$M \boxed{\begin{array}{|c|} \hline \text{X} \\ \hline \end{array}} \quad N \boxed{\begin{array}{|c|} \hline \text{Y} \\ \hline \end{array}} \quad L \boxed{\begin{array}{|c|} \hline \text{Z} \\ \hline \end{array}}$$

$= r \otimes t$

horizontal

Back to 1-categories  $\mathcal{C}$  with  $\otimes$  : standing assumptions  $\Rightarrow$

(identical formulation for 2-categories)

Multi-tensor  
 $C^*$ -category

### $C^*$ structure

$\text{Hom}_{\mathcal{C}}(X, Y)$  are

- $\mathbb{C}$ -vector spaces
- Banach  $\| \cdot \|$
- with antilinear contravariant involution  $t \mapsto t^*$

$$\begin{array}{ccc} X & & Y \\ \downarrow \oplus & \rightarrow & \downarrow \oplus \\ \square & \mapsto & \left( \begin{array}{c|c} X & \\ \hline \square & t \end{array} \right)^* = \begin{array}{c|c} Y & \\ \hline \square & t^* \end{array} \end{array}$$

$$\|sr\| \leq \|s\| \cdot \|r\|$$

$$\|s^*s\| = \|s\|^2$$

### semisimple

$$X, Y \rightsquigarrow X \oplus Y$$

"direct sum"

$X \triangleleft W$  "subobject"

using arrows with the formal properties of isometries  $t^*t = 1_X$

and projections  $p = p^*p$

and every  $X$  is

$$X = \bigoplus_{\text{fin}} X_i$$

$\nwarrow$  "simples"

$$\text{Hom}_{\mathcal{C}}(X_i, X_i) \cong \mathbb{C}$$

### rigid

every  $X$  has a dual  $\bar{X}$   
or conj.  $\bar{\gamma}$   
i.e.  $(\bar{X}, \gamma, \bar{\gamma})$   
"weak inverse"

$$\gamma = \bigcap_{\substack{I \\ X}} \quad , \quad \bar{\gamma} = \bigcap_{\substack{I \\ X}} \bar{X}$$

solving the conj./dual eqns

$$\begin{array}{ccc} X & \rightsquigarrow & \bar{X} \\ \downarrow \gamma & & \downarrow \bar{\gamma} \\ X & = & \bar{X} \end{array} \quad , \quad \begin{array}{ccc} \bar{X} & \rightsquigarrow & X \\ \downarrow \bar{\gamma} & & \downarrow \gamma \\ \bar{X} & = & X \end{array}$$

$\approx "d(X) = d(\bar{X}) < \infty"$

Examples of multi-tensor C\*-categories (and semisimple, rigid 2-C\*-cats)

1)  $N$  properly infinite ,  $\mathcal{C} := \underline{\text{End}}_0(N)$   $\rho(N) \subset N$  finite index  
 $\underline{\text{VN algebra}}$   $\rho, \sigma$  unital \*-endos of  $N$  with  $d(\rho) < \infty$   
 (factor, multifactor)  $\rho \otimes \sigma :=$  composition ,  $t : \rho \rightarrow \sigma$  intertwiners

2)  $N$  finite VN algebra ,  $\mathcal{C} := \underline{\text{Bim}}_0(N)$   $\ell_H(N) \subset R_H(N^{\text{op}})^!$  finite index  
 (factor, multifactor)  $H, K$  Connes'  $N$ - $N$  bimodules, dualizable  
 $H \otimes_N K :=$  Connes fusion ,  $t : H \rightarrow K$  intertwiners

(Similarly :  $\text{VN-Mor}_0$  ,  $\text{VN-Bim}_0$ )

Question: are they all of this form? (for some VN algebras)

Theorem [G, Yuan 20]

(also  $\mathcal{B}$  rigid ssimple 2-C\*-category)

Yes, every  $\mathcal{C}$  multi-tensor C\*-category is realizable

$$F : \mathcal{C} \hookrightarrow \text{Bim}_o(N)$$

$F$  fully faithful unitary tensor functor,  $N$  vN algebra,  $N = \bigoplus_{\text{fin}} N_i$   
(finite multifactor)

Other constructions :  $\text{(*) simple I i.e. } \text{Hom}_{\mathcal{C}}(I, I) \cong \mathbb{C}$

- [Popa 95] : standard lattices (abstract standard invariant)  $\rightsquigarrow$  subfactors
- [Hayashi, Yamagami 00] : amenable C\*-tensor categories on  $N = R$
- [Yamagami 03] : countably generated C\*-tensor categories
- [Brothier, Hartglass, Pennys 12] : countably generated on  $N = \mathbb{C}^{\mathbb{F}_{\infty}}$
- [G, Yuan 19] : C\*-tensor categories on  $N = \mathbb{C}^{\mathbb{F}_{X_1}}$ ,  $N = \prod_{\lambda} \text{factors}$
- [Bischoff, Charlesworth, Evington, G, Pennys 20] : multi-fusion on  $N = \bigoplus_{\text{fin}} R$   
    ↳  $I$  is not necessarily simple  
        ↑  
        "finite depth"

News: bootstrap en OA proof using category theory

def  $X$  in  $\mathcal{C}$  is connected ("irreducibility notion" if  $\mathcal{C}$  not simple)

if  $1_X \otimes \text{Hom}_{\mathcal{C}}(I, I) \cap \text{Hom}_{\mathcal{C}}(I, I) \otimes 1_X = \mathbb{C} \cdot 1_X \left( \subset \text{Hom}_{\mathcal{C}}(X, X) \right)$

$$\left\{ \begin{array}{c|c} X & I \\ \hline \square & \square \end{array} \right\} \cap \left\{ \begin{array}{c|c} I & X \\ \hline \square & \square \end{array} \right\} = \lambda \cdot \begin{array}{c|c} X & X \\ \hline \square & \square \end{array}, \lambda \in \mathbb{C}$$

$X = {}_N H_N$  then connectedness means:  $\ell_H^*(Z(N)) \cap R_H^*(Z(N^{op})) = \mathbb{C} \cdot 1_H$

def a solution  $(\gamma, \bar{\gamma})$  of  $\begin{array}{c|c} X & X \\ \hline \gamma & \gamma \end{array} = \begin{array}{c|c} \bar{X} & \bar{X} \\ \hline \bar{\gamma} & \bar{\gamma} \end{array}$  is standard [Longo, Roberts 97]  
"minimale" [G, Longo 19]

if  $\|\gamma\|^2 = \|\bar{\gamma}\|^2 = \min_{(\eta, \bar{\eta})} \|\eta\| \cdot \|\bar{\eta}\|$  all other solutions of  $\begin{array}{c|c} X & X \\ \hline \gamma & \gamma \end{array} = \begin{array}{c|c} \bar{X} & \bar{X} \\ \hline \bar{\gamma} & \bar{\gamma} \end{array}$  satisfy  $\|\gamma\|^2 = \|\gamma^* \gamma\| = \|\circlearrowleft\|$

- standard solutions describe the minimal index for subfactors  
(category theory)
- when they exist (e.g.  $\mathcal{C}$  semisimple) they are unique

- they come with a <sup>tensor / 2 -</sup>  
 $\sqrt{c}$  categorical "dimension"  $X \mapsto D_X$

$$D_{X \oplus Y} = D_X + D_Y , \quad D_{X \otimes Y} = D_X \cdot D_Y$$

$\uparrow$   
matrix dimension of  $X$

they give "scalar loop parameters": if  $(\bigcap_X, \bigcap_{X \bar{X}})$  is standard

"specialness"  $X$

then 
$$\left| \begin{array}{c} I \\ \bar{x} \bigcirc X \end{array} \right. = \left| \begin{array}{c} I \\ x \bigcirc \bar{x} \end{array} \right. \left| \begin{array}{c} X \\ \bar{X} \end{array} \right. = D_X \cdot \left| \begin{array}{c} X \\ \bar{X} \end{array} \right. , \quad d_X \in [1, \infty[$$

$\uparrow$   
scalar dimension of  $X$

def en algébre in  $\mathcal{C}$  is  $(A, m, \iota)$

some definitions  
for 2-C<sup>\*</sup> categories  $\mathcal{B}$

- $A$  object in  $\mathcal{C}$  (the "algébre object")
- $m : A \otimes A \rightarrow A$  in  $\mathcal{C}$  (the "multiplication")
- $\iota : I \rightarrow A$  in  $\mathcal{C}$  (the "unit")

$\boxed{m} \quad =: \quad \begin{array}{c} A \\ \cup \\ A \end{array}$

$\boxed{\iota} \quad =: \quad \begin{array}{c} \vdots \\ \bullet \\ \downarrow \\ A \end{array}$

such that

$\begin{array}{c} A \\ \cup \\ A \end{array} = \begin{array}{c} \bullet \\ | \\ A \end{array}$

$\begin{array}{c} A & A & A \\ \cup & \cup & \cup \\ A & & A \end{array} = \begin{array}{c} A & A & A \\ \cup & \cup & \cup \\ A & & A \end{array}$

"unit property"

"associativity property"

def we call an algebra  $A$  special if  $m m^* = \begin{array}{c} A \\ | \\ \text{---} \\ | \\ A \end{array} = \begin{array}{c} A \\ | \\ \text{---} \\ | \\ A \end{array} = 1_A$

rmk ≠ terminology from [BKLR15],  
e.g. we demand nothing on  $c^* c = \begin{array}{c} I \\ | \\ \text{---} \\ | \\ I \end{array} \in \text{Hom}_{\mathcal{C}}(I, I)$

lem [LR97], [BKLR15]

$A$  special  $\Rightarrow A$  Frobenius i.e.

$$\begin{array}{c} A \\ | \\ \text{---} \\ | \\ A \end{array} = \begin{array}{c} A \\ | \\ \text{---} \\ | \\ A \end{array} = \begin{array}{c} A \\ | \\ \text{---} \\ | \\ A \end{array}$$

then  $A$  is self-dual in  $\mathcal{C}$ , i.e.  $\bar{A} := A$  with  $(\bigcap_{\bar{A} A}, \bigcap_{A \bar{A}}) := (\begin{array}{c} \bullet \\ | \\ \text{---} \\ | \\ A \end{array}, \begin{array}{c} \bullet \\ | \\ \text{---} \\ | \\ A \end{array})$

lem [BKLR15], [GY20]  $A$  Frobenius  $\Rightarrow A \cong$  special algebras

def a categoricale  $B$ - $A$  bimodule in  $\mathcal{C}$ ,  $A, B$  algebras in  $\mathcal{C}$ ,  $(X, \ell, r)$

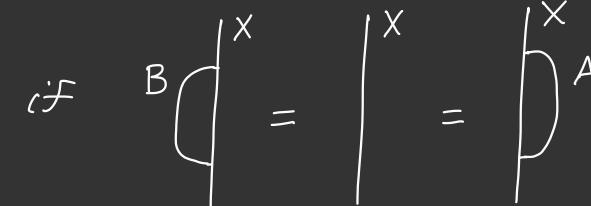
- $X$  object in  $\mathcal{C} \equiv {}_B X_A$
- $\ell : {}_B \otimes X \rightarrow X$  in  $\mathcal{C} = \begin{array}{c} X \\ \text{---} \\ | \\ X \end{array}$  (the "left action of  $B$  on  $X$ ")
- $r : X \otimes A \rightarrow X$  in  $\mathcal{C} = \begin{array}{c} X \\ \text{---} \\ | \\ X \end{array}$  (the "right action of  $A$  on  $X$ )

such that

$$\begin{array}{ccc} {}_B \begin{array}{c} X \\ \text{---} \\ | \\ X \end{array} & = & {}_B \begin{array}{c} X \\ \text{---} \\ | \\ X \end{array}, \quad \begin{array}{c} X \\ \text{---} \\ | \\ A \end{array} = \begin{array}{c} X \\ \text{---} \\ | \\ A \end{array}, \quad \begin{array}{c} X \\ \text{---} \\ | \\ B \end{array} = \begin{array}{c} X \\ \text{---} \\ | \\ A \end{array} \\ \text{"multiplicativity"} & & \text{"unitarity"} \end{array}$$

and left/right actions commute:

$$\begin{array}{c} {}_B \begin{array}{c} X \\ \text{---} \\ | \\ A \end{array} = \begin{array}{c} X \\ \text{---} \\ | \\ A \end{array} =: \begin{array}{c} X \\ \text{---} \\ | \\ A \end{array} \end{array}$$

def as for algebras, we call  ${}_B X_A$  special if 

rmk again ≠ terminology from [BKLR 15]

rmk special  $B$ - $A$  bimodule  $\not\Rightarrow A, B$  special  $ll^* = 1_X = rr^*$

lem [BKLR 15], [GY 20]

${}_B X_A$  bimodule,  $A, B$  special algebras  $\Rightarrow {}_B X_A \cong$  special bimodule

lem [LR 97], [BKLR 15], [GY 20]

$A, B$  special  
 ${}_B X_A$  special }  $\Rightarrow X$  Frobenius i.e.

$$\begin{array}{c} {}^B \curvearrowleft X \\ \curvearrowright {}^B \end{array} = \begin{array}{c} {}^B \curvearrowright X \\ \curvearrowleft {}^B \end{array} = \begin{array}{c} {}^B \curvearrowleft \\ \curvearrowright {}^B \end{array}$$

$$\begin{array}{c} X \curvearrowleft A \\ \curvearrowright A \end{array} = \begin{array}{c} X \curvearrowright A \\ \curvearrowleft A \end{array} = \begin{array}{c} X \curvearrowleft \\ \curvearrowright A \end{array}$$

prop  $s\text{-}\mathcal{Bim}_e(A, B)$  is a  $C^*$ -category  
 ↓↑  
 special      special  
 $(\Rightarrow \text{Frobenius})$        $(\Rightarrow \text{Frobenius})$   
 $B\text{-}A$   
bimodules      algebras

$f: X \xrightarrow{\text{in } \mathcal{C}} X'$   $B\text{-}A$  bimodule intertwiner if

$$\begin{array}{c} X \\ \square f \\ X' \end{array} \quad \text{s.t.}$$

$$\begin{array}{ccc} B & \begin{array}{c} X \\ \square f \\ X' \end{array} & B \\ & = & \\ & \begin{array}{c} X \\ \square f \\ X' \end{array} & \end{array}$$
  

$$\begin{array}{ccc} X & \begin{array}{c} A \\ \square f \\ X' \end{array} & X \\ & = & \\ & \begin{array}{c} X \\ f \square \\ X' \end{array} & \end{array}$$

$f^*: X' \rightarrow X$   $B\text{-}A$  bimodule intertwiner too!

rank: this fails if  $X$  or  $X'$  are not Frobenius bimodules.

$\boxed{s\text{-Bim}_{\mathcal{C}}}$      
 ↗ 0-objects : special algebras in  $\mathcal{C}$  :  $A, B, \dots$  (objects +  $m, c$ )  
 ↗ 1-arrows : special bimodules in  $\mathcal{C}$  :  $X, Y, \dots$  (objects +  $\ell, r$ )  
 ↗ 2-arrows : bimodule intertwiners (arrows commuting with  $\ell, r$ )  
 (same for  $\mathcal{C} \rightsquigarrow \mathcal{B}$ )

vertical composition is the composition of arrows in  $\mathcal{C}$ .

horizontal composition is not just the  $\otimes$  in  $\mathcal{C}$ :

$$\begin{array}{c}
 X \quad \text{B-A bimodule} \\
 Y \quad \text{C-B bimodule}
 \end{array}
 \quad
 \left. \begin{array}{c}
 X \\ \downarrow B \\ A
 \end{array} \right\}
 \quad
 \left. \begin{array}{c}
 Y \\ \downarrow C \\ B
 \end{array} \right\}$$

relative tensor product  
 (of categorical bimodules)

$$\begin{array}{c}
 Y \otimes X \\ \downarrow B \\ C-A \text{ bimodule}
 \end{array}$$

subobject of the "coarse"  
bimodule  $Y \otimes X$  in  $\mathcal{C}$

[Yamagami 04], ...

def of  $Y \otimes_B X$ , let  $\underline{p}_{Y \otimes X}^B := \begin{array}{c} Y \\ \boxed{\text{H}_B} \\ X \end{array} : Y \otimes X \rightarrow Y \otimes X$

then  $\boxed{\text{H}} = \boxed{\text{H}^\bullet} = \boxed{\text{H}^\circ} = \boxed{\text{H}}$ , in particular  $\boxed{\text{H}^\bullet} = \boxed{\text{H}^\circ} = : \boxed{\text{H}^\bullet} \boxed{\text{H}^\circ} : \Rightarrow p \text{ is self-adjoint}$

$$\boxed{h} = \boxed{h^\bullet} = \boxed{h^\circ}$$

also  $\boxed{H} = \boxed{H^\bullet} = \boxed{\text{H}^\bullet \text{ (dashed)}} = \boxed{H^\circ} = \boxed{\text{H}^\circ \text{ (dashed)}} = \boxed{h^\bullet} = \boxed{h^\circ} \Rightarrow p \text{ is idempotent}$

$p = p^* p$  in  $\text{Hom}_\ell(Y \otimes X, Y \otimes X)$   $\longleftrightarrow$  subobject of  $Y \otimes X$

let

$$\begin{array}{c} Y \quad X \\ | \quad | \\ \boxed{B} \end{array} = \begin{array}{c} Y \quad X \\ \downarrow \quad \downarrow \\ \text{subobject of } Y \otimes X \text{ denoted by } \underline{Y \otimes_B X} \in s\text{-}\mathbf{Bim}_{\mathcal{C}}(A, C) \\ (\text{unique up to unitaries}) \\ \uparrow \quad \uparrow \\ \text{isometry } S \text{ s.t. } SS^* = P_{Y \otimes X}^B \end{array}$$

$$\begin{array}{c} Y \otimes_B X \\ A \\ \vdash \end{array} := \begin{array}{c} Y \otimes_X X \\ A \\ \vdash \\ Y \otimes_B X \end{array}$$

moreover

$$\begin{array}{c} Y \quad X \\ f \blacksquare \quad g \\ | \quad | \\ Y' \quad X' \end{array} = \begin{array}{c} Y \quad X \\ f \blacksquare \quad g \\ | \quad | \\ Y' \quad X' \end{array}, \text{ denote } \underline{f \otimes_B g} := \begin{array}{c} Y \otimes_X X \\ f \blacksquare \quad g \\ \vdash \\ Y' \otimes_B X' \end{array}$$

and  $f \otimes_B g \in s\text{-}\mathbf{Bim}_{\mathcal{C}}(A, C)(Y \otimes_B X, Y' \otimes_B X')$

Then  $s\text{-Bim}_{\mathcal{C}}$  is a  $C^*$ -bicategory (non-strict) (same for  $\mathcal{C} \otimes \mathcal{B}$ )

- horizontal units  $I_A := A$  with  ${}^A \downarrow^A = \downarrow^A := \begin{array}{c} A \\ \sqcup \\ A \end{array}$  ( $I_A$  special if  $A$  special)
- rigid ( $X$  dualizable in  $\mathcal{C} \Rightarrow X$  dualizable  $B$ - $A$  bimodule [Yam04])
- semisimple (bimodule intertwiners  $\subset \text{Hom}_{\mathcal{C}}(X, X')$  are finite-dim)

moreover there is an embedding:  $\boxed{\mathcal{C} \hookrightarrow s\text{-Bim}_{\mathcal{C}}}$

$X \mapsto {}_I X_I$  special  $I$ - $I$  bimodule,  $I$  with trivial algebra structure

- $Y \otimes_I X = Y \otimes X$  because  $\begin{array}{c} Y \times \\ | \sqcup | \\ I \end{array} = \begin{array}{c} Y \times \\ | \\ | \end{array}$  (strictly tensor)

- $s\text{-Bim}_{\mathcal{C}}(I, I)(X, X') = \text{Hom}_{\mathcal{C}}(X, X')$  (fully faithful)

Thm [GY20] for every (indecomposable) multi-tensor  $C^*$ -category  $\mathcal{C}$

then  $\mathcal{C} \simeq s\text{-}\mathrm{Bim}_{\mathcal{C}_0}(A, A)$ ,  $\mathcal{C}_0$  tensor  $C^*$ -category

$A$  special algebra in  $\mathcal{C}_0$



even "standard" and  
"canonical"

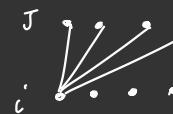
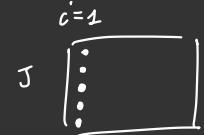
$$(A = \bar{X} \otimes X)$$

$\mathcal{C}_0 := \mathcal{C}_{11}$ ,  $A$  := using standard solutions

pF(sKetchn)

let  $I = \bigoplus_{i=1}^m I_i$ ,  $I_i$  simples in  $\mathcal{C}$ ,

for every  $j = 1, \dots, m$ , let  $X_j = I_j \otimes X_j \otimes I_1 \neq 0$  and  $\underline{X} := \bigoplus_{j=1}^m X_j$  in  $\mathcal{C}$



$X \otimes I_i \neq 0$  iff  $i=1$        $\left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow X$  is connected, choose std solutions  $\bigcap_{\bar{x}} X, \bigcap_{X} \bar{x}$   
 $I_j \otimes X \neq 0 \quad \forall j = 1, \dots, m$        $\left. \begin{array}{l} \\ \\ \end{array} \right\} \text{and } \times \bigodot^{\bar{x}} |^X = d_X \cdot |^X \Rightarrow \times \bigodot^{\bar{x}} = d_X \bigcap_I$

$$\text{set } \underline{\underline{A}} := (\bar{x} \otimes x, \quad \begin{array}{c} A \\ \cup \\ A \end{array} := \frac{1}{\sqrt{dx}} \cdot \begin{array}{c} \bar{x} & x & \bar{x} & x \\ | & \cup & | & | \\ \bar{x} & x & x & x \end{array}, \quad \begin{array}{c} \bullet \\ A \end{array} := \sqrt{dx} \cdot \begin{array}{c} \cap \\ \bar{x} & x \end{array})$$

special  
standard  
canonical

$A$  is a special algebra

$$\begin{array}{c} A \\ \circ \\ A \end{array} = \frac{1}{\sqrt{dx}} \cdot \begin{array}{c} \bar{x} \\ | \\ \circ \end{array}^x = \begin{array}{c} \bar{x} & x \\ | & | \end{array} = \begin{array}{c} A \\ | \\ A \end{array} \quad \text{by } x \circ \bar{x} = dx$$

$$\begin{array}{c} A \\ \cup \\ A \\ \cup \\ A \\ A \end{array} = \frac{1}{\sqrt{dx}} \cdot \begin{array}{c} \bar{x} & x & \bar{x} & x & \bar{x} & x \\ | & \cup & | & \cup & | & \cup \\ \bar{x} & x & x & x & x & x \end{array} = \frac{1}{\sqrt{dx}} \cdot \begin{array}{c} \bar{x} & x & \bar{x} & x & \bar{x} & x \\ | & \cup & | & \cup & | & \cup \\ \bar{x} & x & x & x & x & x \end{array} = \begin{array}{c} A \\ \cup \\ A \\ \cup \\ A \\ A \end{array}, \quad \text{associative } \checkmark$$

$$\begin{array}{c} \bullet \\ A \\ | \\ A \end{array} = \begin{array}{c} \bar{x} & x \\ | & | \\ \bar{x} & x \end{array} = \begin{array}{c} \bar{x} & x \\ || & | \\ \bar{x} & x \end{array} = \begin{array}{c} A \\ | \\ A \end{array} \quad \text{unit } \checkmark$$

$$A \in \mathcal{C}_{11}, \text{ as } I_J \otimes A \otimes I_K = I_J \otimes \bar{x} \otimes x \otimes I_K = \begin{array}{c} A \\ \circ \\ \circ \end{array} \quad \begin{array}{l} \text{if } J=K=1 \\ \text{otherwise} \end{array}$$

$$F: \mathcal{C} \longrightarrow S\text{-}\mathrm{Bim}_\varphi(A, A) \quad , \quad A = \bar{x} \otimes x$$

$$Y \longmapsto \bar{x} \otimes Y \otimes x = \begin{array}{c} \bar{x} \\ | \\ Y \\ | \\ x \end{array} \quad \text{with} \quad \begin{array}{c} \bar{x}x \quad \bar{x}Yx \quad \bar{x}x \\ \swarrow \quad \downarrow \quad \searrow \\ \bar{x}Yx \end{array} := \frac{1}{d_x} \cdot \begin{array}{c} \bar{x}x\bar{x} \\ \swarrow \quad \downarrow \quad \searrow \\ Y \\ \swarrow \quad \searrow \\ x\bar{x}x \end{array}$$

*A-A bimodule*

clearly  $\Psi = \psi$

special by  $\frac{1}{d_x^2} \cdot \left\langle \begin{array}{|c|c|} \hline \circ & \circ \\ \hline \end{array} \right\rangle = \begin{array}{|c|c|c|} \hline \circ & \circ & \circ \\ \hline \end{array}$

$$\tau: Y \rightarrow Y' \longmapsto 1_{\bar{x}} \otimes \tau \otimes 1_x = \begin{array}{c} \bar{x} \\ | \\ Y \\ | \\ \tau \\ | \\ Y' \end{array} \quad \text{A-A bimodule intertwiner } \Psi = \psi$$

check:

multiplication  $\Psi = \begin{array}{c} \cup \\ \backslash \\ \parallel \end{array} = \begin{array}{c} \cup \\ \backslash \\ \parallel \end{array} \parallel = \begin{array}{c} \cup \\ \cup \\ \parallel \end{array} = \begin{array}{c} \cup \\ \parallel \end{array} \quad , \quad \text{unit} \quad \psi = \begin{array}{c} \cup \\ \parallel \end{array} = \begin{array}{c} \cup \\ \parallel \end{array} = \begin{array}{c} \parallel \end{array}$

$A = \bar{X} \otimes X$  special

ess 54:  $W \in s\text{-Bim}_\mathbb{C}(A, A)$ , then  $W \cong \begin{array}{|c|c|c|} \hline \bar{x} & y & x \\ \hline | & | & | \\ \hline \end{array}$

↑  
tensor, simple in [BKLR15]

unitarily as  $A$ - $A$  bimodule

using tensoring on the left:  $\begin{array}{|c|} \hline \bar{x} \\ \hline | \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline \bar{x} & \\ \hline | & | \\ \hline \end{array}$  is injective by  $x \odot \bar{x} = d_x$   
(not only  $\odot \begin{array}{|c|} \hline x \\ \hline | \\ \hline \end{array} = d_x \cdot \begin{array}{|c|} \hline x \\ \hline | \\ \hline \end{array}$ )

full:  $\begin{array}{|c|c|c|} \hline \bar{x} & y & x \\ \hline | & | & | \\ \hline \bar{x} & y' & x \\ \hline \end{array}$   $A$ - $A$  bimodule intertwiner, then  $\begin{array}{|c|c|c|} \hline \bar{x} & y & x \\ \hline | & | & | \\ \hline \bar{x} & y' & x \\ \hline \end{array} \stackrel{!}{=} \begin{array}{|c|c|} \hline \odot \begin{array}{|c|} \hline \bar{x} \\ \hline | \\ \hline \end{array} & \\ \hline | & | \\ \hline \end{array} =: \begin{array}{|c|c|} \hline t & \\ \hline | & | \\ \hline \end{array}$

faith:  $\begin{array}{|c|c|c|} \hline \bar{x} & y & x \\ \hline | & | & | \\ \hline \bar{x} & y' & x \\ \hline \end{array} = 0 \Rightarrow x \odot \bar{x} \begin{array}{|c|c|} \hline y & \\ \hline | & | \\ \hline y' & \\ \hline \end{array} = d_x^2 \cdot \begin{array}{|c|c|} \hline y & \\ \hline | & | \\ \hline y' & \\ \hline \end{array} = 0$

tensor:  $F(Y) \otimes_A F(Z) = \begin{array}{|c|c|c|} \hline \bar{x} & Y & X \\ \hline | & | & | \\ \hline \bar{x} \otimes X & & \end{array} \otimes \begin{array}{|c|c|c|} \hline \bar{x} & z & X \\ \hline | & | & | \\ \hline \bar{x} \otimes X & & \end{array} \leq F(Y) \otimes F(Z)$

$$\begin{array}{ccc} \begin{array}{|c|c|c|} \hline \bar{x} & Y & X \\ \hline | & | & | \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \bar{x} & z & X \\ \hline | & | & | \\ \hline \bar{x} \otimes X & & \end{array} & = & \frac{1}{dx} \cdot \begin{array}{|c|c|c|} \hline \bar{x} & Y & X \\ \hline | & | & | \\ \hline \bar{x} & z & X \\ \hline | & | & | \\ \hline \bar{x} \otimes X & & \end{array} \\ \text{P}_A^A = \\ F(Y, FZ) & & \end{array} = \frac{1}{dx} \cdot \begin{array}{|c|c|c|} \hline \bar{x} & Y & X \\ \hline | & | & | \\ \hline \cup & & | \\ \hline \cap & & | \\ \hline \end{array} = \frac{\frac{1}{dx} \cdot \left( \begin{array}{c} \cup \\ \cap \end{array} \right)}{\frac{1}{\sqrt{dx}} \cdot \left( \begin{array}{c} \cap \\ \cup \end{array} \right)}$$

$\approx ss^*$

$$\frac{1}{dx} \cdot \begin{array}{|c|c|c|} \hline \bar{x} & Y & z & X \\ \hline | & | & | & | \\ \hline \circ & & & | \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \bar{x} & Y & z & X \\ \hline | & | & | & | \\ \hline & & & | \\ \hline \end{array} = F(Y \otimes Z)$$

unitary, unital :  $F(t^*) = F(t)^*$ ,  $F(I) = A$  trivial  $A$ - $A$  bimodule  $\square$

$$F : \mathcal{C} \xrightarrow{\sim} \text{s-Bim}_\mathcal{C}(A, A) , \quad A = I_1 \otimes A \otimes I_1 \in \mathcal{C}_{11}^{\text{obj}} \quad A = \begin{array}{|c|c|} \hline \bar{x} & x \\ \hline \end{array}$$

$\mathcal{C}_{11}$

$$F(Y) = I_1 \otimes F(Y) \otimes I_1 \in \mathcal{C}_{11}^{\text{wr}}$$

$$F(Y) = \begin{array}{|c|c|c|} \hline \bar{x} & y & x \\ \hline \end{array}$$

using this, end tensor realization results results :

$$\mathcal{C}_{11} \hookrightarrow \underline{\text{vN-Bim}_0}(N) , \quad N \text{ II}_1 \text{ factor}$$

$$A \mapsto \underbrace{\text{vN-algebraic special (std) } C^*\text{-Frob algebras}}_{\text{Q-system}} \quad \xleftarrow{\text{Fun}} \quad N \underset{\cong}{\subset} M \quad \text{II}_1 \text{ vN alg}$$

$\begin{array}{c} \text{factor} \\ \iff \\ A \text{ simple} \end{array}$

$\sim [\text{BIRLR15}]$   
in  $\text{End}_0(N)$

then  $\boxed{\mathcal{C} \hookrightarrow \text{vN-Bim}_0(M)} , \quad M = \bigoplus_{i=1}^m M_i : \text{II}_1 \text{ factors}$

Thank You !