

QSSS Talk, OSU

FUSION STRUCTURE FROM EXCHANGE SYMMETRY IN (2+1)-DIMENSIONS

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INTRO

- "Quantum symmetries" \sim "Topological phases of matter"
- Why are anyons described by URFCs / MTCs?
- Using a minimal prescription of postulates (which we'll highlight in yellow),
Can we derive URFCs as the algebraic framework for anyons?

Idea ("Exchange statistics")
 Permute n identical particles in \mathbb{R}^d . How does the state of the system evolve? ($d \geq 2$)

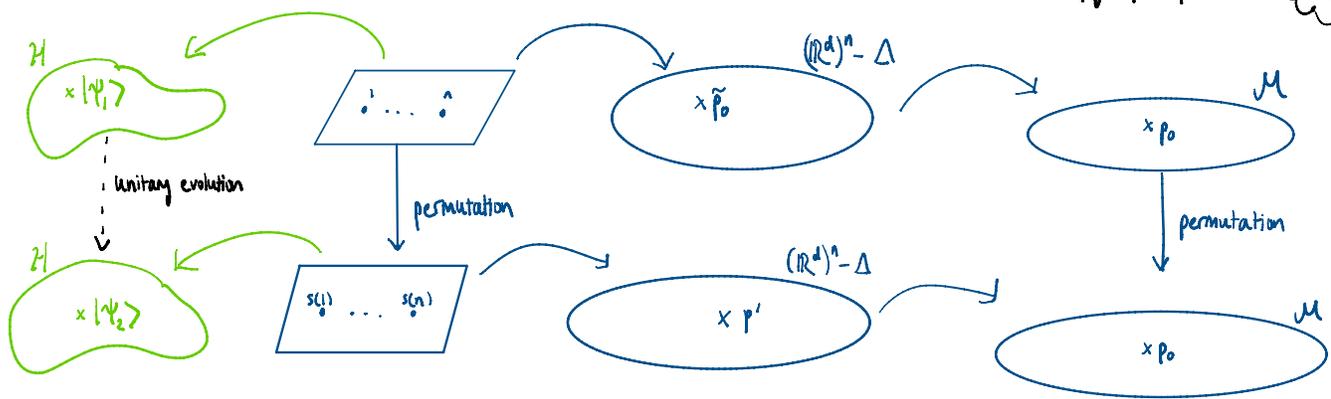
- Identical particles have the same intrinsic properties (spin, mass, charge etc.): cannot distinguish between them!
 E.g. all electrons in the universe are identical.

- What's the configuration space \mathcal{M} of n identical particles in \mathbb{R}^d ?
 - Let $\Delta := \{ (z_1, \dots, z_n) \in (\mathbb{R}^d)^n : z_i = z_j \text{ for some } i \neq j \}$.
 - Want $(\mathbb{R}^d)^n - \Delta$
 - Particles identical \therefore identify $\{ (z_{s(1)}, \dots, z_{s(n)}) \}_{s \in S_n}$.
 This gives an equivalence relation " \sim " on $(\mathbb{R}^d)^n - \Delta$

UPSHOT

$$\mathcal{M} = \frac{(\mathbb{R}^d)^n - \Delta}{\sim}$$

" n^{th} unordered config. sp. of \mathbb{R}^d " $\mathcal{U}_n(\mathbb{R}^d)$



Let $\Pi(p_0, p_0)$ be the space of all loops in \mathcal{M} (w/ base point p_0) i.e. "set of exchange trajectories for n identical particles"

The homotopy classes of paths in $\Pi(p_0, p_0)$ are given by $\pi_1(\mathcal{M})$. (\therefore write $\Pi(p_0, p_0) = \bigsqcup_{g \in \pi_1(\mathcal{M})} [g]$)

Idea from QFT (Feynman path integral formulation) (fundamental gr)

Evolution of a state along trajectory $\gamma \in [g] \in \Pi(p_0, p_0)$ where $g \in \pi_1(\mathcal{M})$ is $\rho(g)$
 where

$$\rho: \pi_1(\mathcal{M}) \rightarrow U(\mathcal{H}) \quad \text{(unitary linear representation)}$$

FACTS

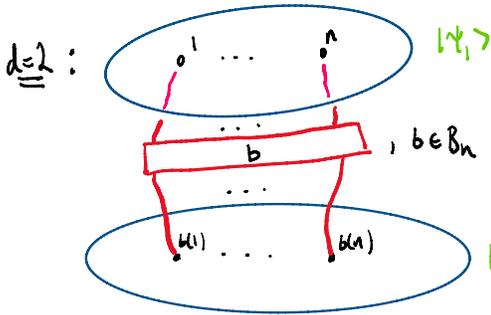
(1) $\pi_1(\mathcal{M}) \cong B_n, d=2$

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2 \end{array} \right\rangle$$

(2) $\pi_1(\mathcal{M}) \cong S_n, d \geq 3$

$$S_n = \left\langle s_1, \dots, s_{n-1} \mid \begin{array}{l} s_i^2 = e \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i, |i-j| \geq 2 \end{array} \right\rangle$$

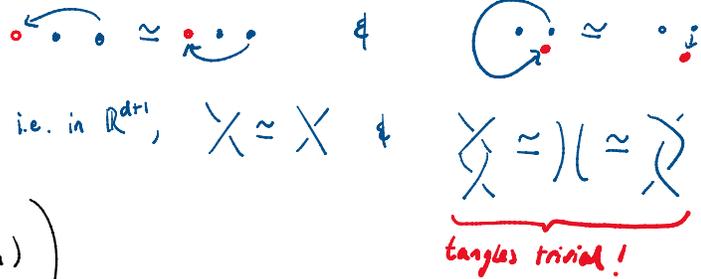
• This translates to an intuitive picture.



$|\psi_2\rangle = \rho(b) |\psi_1\rangle, \rho: B_n \rightarrow U(\mathcal{H})$

- Strands have same colour \therefore particles identical
- Trajectories in \mathbb{R}^{d+1} : "worldlines" / "histories"

$d \geq 3$: For a pairwise exchange in \mathbb{R}^d ,



(Corresponds to, $\eta: B_n \rightarrow S_n$ (ker $\eta = PB_n$)
 $\sigma_i^{-1} \mapsto s_i$)

Remark: By Alexander's th^m, this recovers the fact that there are no nontrivial links in ≥ 3 dimensions!

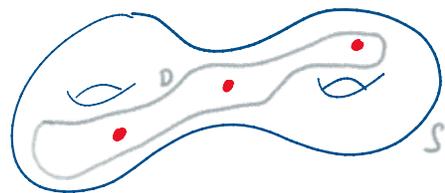
So here, $|\psi_2\rangle = \rho(s) |\psi_1\rangle, \rho: S_n \rightarrow U(\mathcal{H})$

• Note

- For a surface S , $\pi_1(\mathcal{U}_n(S)) \cong B_n(S)$ (surface braid group)

- Inclusion $D \hookrightarrow S$ induces homomorphism $\iota: B_n(D) \rightarrow B_n(S)$
 where $B_n(D) \cong B_n$

\therefore Artin braid relations also hold in $B_n(S)$



- **Consequence**: What we learn about particle exchanges on a disc will also hold true on surface S

} so we'll always consider particles in a disc for $d=2$.

• OK, so exchange statistics:

(1) $d \geq 3$: $\rho: S_n \rightarrow U(\mathcal{H})$

$\dim(\mathcal{H}) = 1$

Either have $\rho = \rho^+$ or $\rho = \rho^-$ where $\rho^+(s_i) = 1 \forall i$
 $\rho^-(s_i) = -1 \forall i$

- Doplicher, Haag, Roberts (70s): You can

ρ^+ : BOSONS, ρ^- : FERMIONS

$$p^-(0,1) = -1 \quad \forall i$$

- Doplicher, Haag, Roberts (70s): You can only get bosons/fermions for $d \geq 3$
- Müger (07): "

p^+ : BOSONS, p^- : FERMIONS

$$\underline{\dim(\mathcal{H}) > 1}$$

Higher-dim. reps: "PARASTATISTICS"



\therefore All fundamental particles may be classified as bosons/fermions

(2) $d=2$: $\rho: B_n \rightarrow U(1)$

$$\underline{\dim(\mathcal{H}) = 1}$$

$$\rho: B_n \rightarrow U(1)$$

$$\sigma_i \mapsto e^{i\theta}$$

"ANYONS" (Wilczek, '82)

"fractional statistics"

- No fundamental particles in 2D.
 - Anyons are **QUASIPARTICLES**
-
- Anyons emerge from microsystems.
- \hookrightarrow Spatially localized properties of 2D system
 - \hookrightarrow No internal degrees of freedom \therefore identical

$$\underline{\dim(\mathcal{H}) > 1}$$

Higher-dim. reps:

"non-abelian anyons"



OK, but how do I get from fractional statistics to ribbon categories?!

Superselection Sectors

- Given a quantum system w/ Hilbert space \mathcal{H} , a **SUPERSELECTION RULE (SSR)** is given by a linear operator $\hat{J}: \mathcal{H} \rightarrow \mathcal{H}$ s.t.

$$\boxed{[\hat{O}, \hat{J}] = 0 \quad \text{for all observables } \hat{O}}$$

\hookrightarrow Let \mathcal{H}' & \mathcal{H}'' be 2 distinct eigenspaces of \hat{J} . Write, $\hat{J}|\psi'\rangle = z'|\psi'\rangle$, $z' \in \mathbb{C}$
 $\hat{J}|\psi''\rangle = z''|\psi''\rangle$, $z'' \in \mathbb{C}$

Then, $z' \langle \psi'' | \hat{O} | \psi' \rangle = \langle \psi'' | \hat{O} \hat{J} | \psi' \rangle = \langle \psi'' | \hat{J} \hat{O} | \psi' \rangle = z'' \langle \psi'' | \hat{O} | \psi' \rangle$

$$\therefore \boxed{\langle \psi'' | \hat{O} | \psi' \rangle = 0}$$

Consequence: $\langle \psi | \hat{O} | \psi \rangle = \langle \psi_\theta | \hat{O} | \psi_\theta \rangle \quad \forall \hat{O}, \theta$, $|\psi\rangle := \alpha |\psi'\rangle + \beta |\psi''\rangle$

(normalised states)

Consequence: $\langle \Psi | \hat{O} | \Psi \rangle = \langle \Psi_0 | \hat{O} | \Psi_0 \rangle \quad \forall \hat{O}, \theta$, $|\Psi\rangle := \alpha |\Psi'\rangle + \beta |\Psi''\rangle$ (normalised states)
 $|\Psi_0\rangle := \alpha |\Psi'\rangle + e^{i\theta} \beta |\Psi''\rangle$

i.e. cannot observe relative phases \therefore can't distinguish between superpositions and statistical mixtures!

⇒ Cannot observe superpositions over eigenspaces of \hat{J} !

Terminology : Eigenspaces of \hat{J} called SUPERSELECTION SECTORS.

- Examples of SSRs : (1) Spin , (2) Electric charge , (3) In limit $c \rightarrow \infty$: mass

Spin-Statistics Theorem (Pauli) : Bosons have integer spin
 Fermions have half-integer spin

↑
 (1) Boson-fermion SSR

Intrinsic properties of a particle
 ||
 Quantum numbers w/ an associated SSR

Exchange Symmetry & Superselection Sectors

- For a system of n particles, exchange symmetry may be concisely expressed by

$$[\hat{O}, \rho(g)] = 0$$

\Leftrightarrow

$$[\hat{O}, \rho(g_i)] = 0 \quad \forall i$$

for all observables \hat{O} on Hilbert space \mathcal{H}
 for all $g \in G = \begin{cases} B_n, d=2 \\ S_n, d \geq 3 \end{cases}$
 $\rho: G \rightarrow U(\mathcal{H})$

$G = \langle g_i \rangle_i$

(i.e. eigenspaces of \hat{O} stable under action of G \therefore measurement of \hat{O} unaffected by permutations)

Consequence: Eigenspaces of $\rho(g_i)$ are superselection sectors!

- Notice : $\text{Spectrum of } \rho(\sigma_i) \subseteq U(1)$
 $\text{" " } \rho(s_i) \subseteq \{\pm 1\}$

Remark

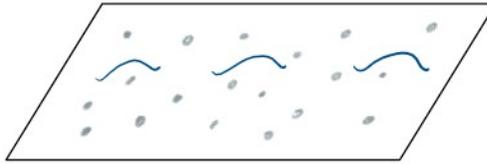
- (i) Boson-fermion SSR / symmetrisation postulate / work of DHR & Müger lay constraints on eigenspaces of $\rho(s_i)$

The key is spatial localisation

- (ii) When $d=2$, how do we make this picture consistent w/ non-abelian statistics and the idea of distinct mutual statistics between different pairs of anyons?

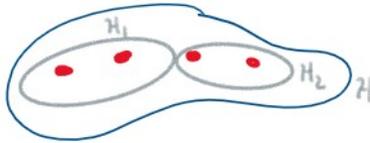
Anyons emerge as spatially localized phenomena in 2D

So we can concretely talk about transporting/exchanging them



↳ Also means that we can consider subsystems of particles!

(Compare this to Fock space for bosons / fermions ; $\mathcal{H}_{\pm} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)} \oplus \dots$
(e.g. $\mathcal{H}_+^{(2)} \neq \mathcal{H}_+^{(3)}$)



We can consider the Hilbert spaces associated to subsystems of anyons by spatial localisation

Exchange Statistics under Spatial Localisation

• The idea : "Exchange symmetry for d=2" + "spatially localized quasiparticles" = ?

• 2-quasiparticle system w/ Hilbert space \mathcal{D}

$$\rho: B_2 \rightarrow U(\mathcal{D})$$

$$\mathcal{D} = \bigoplus_x \mathcal{D}_x \text{ where } \mathcal{D}_x \text{ are superselection sectors } \& \rho(\sigma_i)|\psi\rangle = e^{i\alpha_x}|\psi\rangle, |\psi\rangle \in \mathcal{D}_x$$

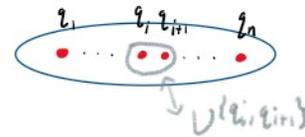


Remark

Assume index set of x is finite and $\dim(\mathcal{D}_x) < \infty$

↳ Suppose 2 quasiparticles are subsystem of n -particle system.

By localisation, the superselection sectors / exchange statistics are an intrinsic property of the pair of particles (indep. of rest of system)



So we write,

$$\rho\{\{i, j\}\} : B_2 \rightarrow U(\mathcal{D}^{\{i, j\}}) \text{ , decomposition } \mathcal{D}^{\{i, j\}} = \bigoplus_x \mathcal{D}_x^{\{i, j\}} \text{ under } \rho\{\{i, j\}\}(\sigma_i)$$

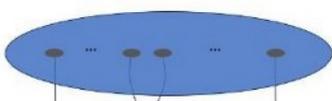
(unordered set!)

• n -quasiparticle Hilbert spaces : - Label quasiparticles $1, 2, \dots, n$.

- Let $S_{\{1, \dots, n\}}$ denote set of all permutations of string $12 \dots n$

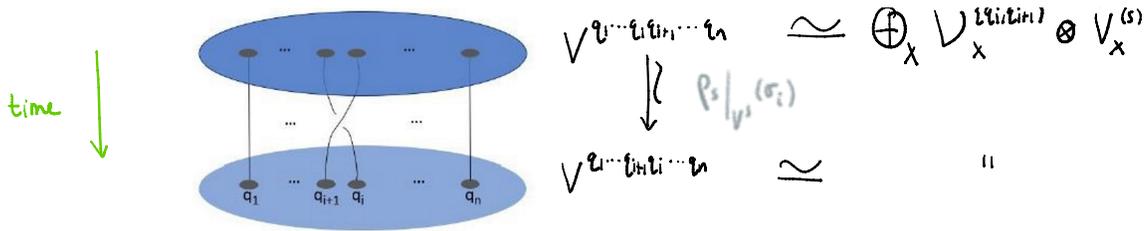
- Write $s = q_1 \dots q_n$ where q_i is i th character of $s \in S_{\{1, \dots, n\}}$

- Denote Hilbert space for quasiparticles $q_1 \dots q_n$ (in that order) by $\mathcal{V}^{\{q_1 \dots q_n\}}$ (or \mathcal{V}^s)



$$\mathcal{V}^{\{q_1 \dots q_n\}} \cong \bigoplus_x \mathcal{D}_x^{\{q_1 \dots q_n\}} \otimes \bar{\mathcal{V}}_x^{(s)}$$

$| \rho_s |(\sigma_i)$



The clockwise exchange of quasiparticles q_i and q_{i+1} .

($\bar{V}_X^{(s)}$ denotes state space of rest of system when $q_i \neq q_{i+1}$ are in superselection sector X)

We have,

$$P_S|_{V_X}(\sigma_i^{\pm 1}) = \bigoplus_X \left[P_{\{q_i, q_{i+1}\}}^X(\sigma_i^{\pm 1}) \otimes \text{id}_{\bar{V}_X^{(s)}} \right], \quad P_{\{q_i, q_{i+1}\}}^X \text{ is subrep } P_{\{q_i, q_{i+1}\}}|_{V_X^{q_i, q_{i+1}}} \quad (*)$$

• Definition

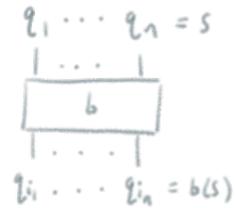
$P_S(\sigma_i^{\pm 1})$ denotes the action of (anti)clockwise exchanging $q_i \neq q_{i+1}$ on some V^u , $u \in S_{\{1, \dots, n\}}$

↳ following $(*)$, we have $P_S|_{V^u}(\sigma_i^{\pm 1}) = \bigoplus_X \left[P_{\{q_i, q_{i+1}\}}^X(\sigma_i^{\pm 1}) \otimes \text{id}_{\bar{V}_X^{(u)}} for any u containing the substring $q_i q_{i+1}$ or $q_{i+1} q_i$$

↳ Note: E.g. action of $\sigma_i \in B_3$ on V^{123} will differ from that on V^{321} (∴ we distinguish between $\{V^s\}_s$ to consider action of braiding on whole system.)

• Notation

Given $s \in S_{\{1, \dots, n\}}$ & $b \in B_n$, let $b(s)$ denote the "obvious" permutation of s



• The action of braiding on an n -quasiparticle system is given by linear transformations between spaces $\{V_s\}_s$ defined by functions $\{P_S\}_s$ s.t.

(0) Domain of P_S is B_n

(1) $P_S(b) : \bigoplus_{u \in \mathcal{U}_{S,b}} V^u \rightarrow \bigoplus_{s' \in S_{\{1, \dots, n\}}} V^{s'}$ is a linear transformation, $\mathcal{U}_{S,b} \subseteq S_{\{1, \dots, n\}}$

(2) For any $u \in \mathcal{U}_{S,b}$, have unitary linear iso. $P_S|_{V^u}(b) : V^u \xrightarrow{\cong} V^{b(u)}$ and if $u' \notin \mathcal{U}_{S,b}$ then $P_S(b)$ is undefined on $V^{u'}$

(3) For $b = b_2 b_1$, have $P_S|_{V^u}(b_2 b_1) = P_{b_1(s)}|_{V^{b_1(u)}}(b_2) \circ P_S|_{V^u}(b_1)$, $u \in \mathcal{U}_{S,b_1}$ & $b_1(u) \in \mathcal{U}_{b_1(s), b_2}$

(4) $P_S|_{V^u}(\sigma_i^{\pm 1})$ is as defined above.

THINK: $\{P_s\}_s$ is unitary linear repⁿ of coloured braid groupoid

Equivalent to some functor $Z: B_n(\mathcal{B}) \rightarrow \text{FdHilb}$ (Image of morphisms are unitary transⁿ), $\mathcal{B} := S_{\{1, \dots, n\}}$

- Well-definedness of $\{P_s\}_s$? From (3) above, have to ensure that all distinct ways to parse $P_s|_{\nu}(b)$ into generators agree ("coherence") - fulfilled by hexagon eqs!

Exchange Symmetry for n Quasiparticles

- We understand E.S. for 2 quasiparticles. What about $n > 2$?

↳ Naive attempt: $[\hat{O}, P_s(b)] = 0, \forall s \in S_{\{1, \dots, n\}}, b \in B_n$ } ill-defined!

↳ Puzzle

Given $\{V^s\}_{s \in S_{\{1, \dots, n\}}}$, can we recover the n -particle Hilbert space "modulo ordering"?

Denote this space by $V^{[n]}$ where $[n] := \{1, \dots, n\}$ is unordered.

Sketch of solⁿ

- Take $E_n \subseteq B_n$ s.t. for all $g \in E_n$, eigenspaces of g are stable under action of B_n i.e. for eigenspace decomp. $V^S = \bigoplus_Q V_Q^S$ under $P_s(g)$, we have

$$P_{b(s)}(g) \cdot P_s(b) |\psi\rangle = e^{i\mu_Q} |\psi\rangle$$

where $P_s(g) |\psi\rangle = e^{i\mu_Q} |\psi\rangle$ for any $|\psi\rangle \in V_Q^S$.

∴ V_Q^S is $e^{i\mu_Q}$ -eigenspace of $P_s(g)$ for all $s \in S_{\{1, \dots, n\}}$, so denote by $V_Q^{[n]}$

- Write $V^{[n]} = \bigoplus_Q V_Q^{[n]}$ & $P^{[n]}(g) = \sum_Q e^{i\mu_Q} \hat{P}_Q$, \hat{P}_Q is orthogonal projector onto $V_Q^{[n]}$
 $P^{[n]}: E_n \rightarrow U(V^{[n]})$

Theorem

$$E_n = \langle \underline{P}_n \rangle \subseteq B_n$$

i.e. $\langle \underline{P}_n \rangle$ describes all braids whose eigenspaces are preferred under the action of B_n

↳ So we can write $[P^{[n]}(g), \hat{O}] = 0 \quad \forall g \in E_n$ and $\forall \hat{O}$ on $V^{[n]}$

n -quasiparticle exchange symmetry

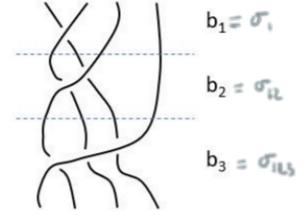
↳ S. superselection sectors in systems are distinguished at R . (∴ R called SUPERSELECTION BRAID)

→ So we can write $[P_{\text{ex}}(g), O] = 0 \quad \forall g \in E_n \text{ and } \forall O \text{ on } V^{\otimes n}$ n-quasiparticle exchange symmetry

↳ Superselection sectors of system are eigenspaces of β_n ($\therefore \beta_n$ called **SUPERSELECTION BRAID**)

↳ This shows that exchange symmetry for n particles emerges from "local exchange symmetries" for pairs of constituent particles!

• What is β_n ? Heuristics: - Symmetry
- Does not favour any one particle over another



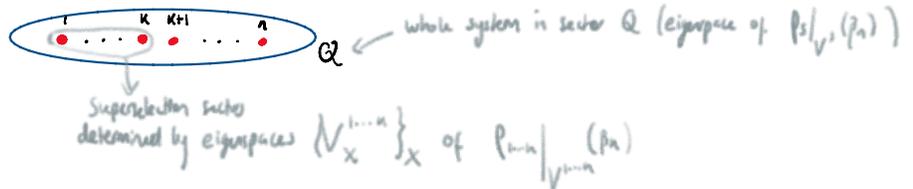
$$\beta_n = \sigma_{12} \sigma_{13} \dots \sigma_{1n} \sigma_{23} \dots \sigma_{2n} \dots \sigma_{(n-1)n} \sigma_{12} \sigma_{13} \dots \sigma_{1n} \sigma_{23} \dots \sigma_{2n} \dots \sigma_{(n-1)n}$$

β_n has length $\binom{n}{2}$. The above diagram depicts β_4 . (Exchanges each pair once)

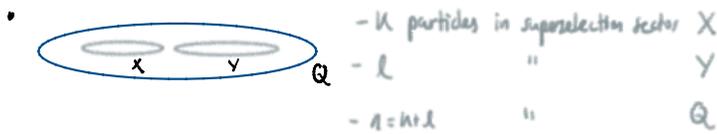
↳ Satisfies several internal symmetries!

Hierarchy of Superselection Sectors

• Given n quasiparticles, have an exchange symmetry mechanisms local to any k adjacent quasiparticles



Decomposition: $V_Q^{i_1, \dots, i_n} = \bigoplus_X V_X^{i_1, \dots, i_n} \otimes V_Q^{x, i_{k+1}, \dots, i_n}, \quad V_Q^{i_1, \dots, i_n} = \bigoplus_Q V_Q^{i_1, \dots, i_n}$



Decomposition: $V_Q^{i_1, \dots, i_n} = \bigoplus_{x,y} V_X^{i_1, \dots, i_n} \otimes V_Q^{xy} \otimes V_Y^{i_{k+1}, \dots, i_n}$

↳ Spaces $\{V_Q^{xy}\}_{x,y}$ "constrain" subsystems w.r.t. Q.

↳ Sector X & Y "contained in Q in $\dim(V_Q^{xy})$ distinct ways" (could be 0)

• Remark (superposition)

Local symmetry mechanisms superseded by global ones i.e. $\{V_X^{i_1, \dots, i_n}\}_X$ are superselection sectors w.r.t. n-particle subsystems

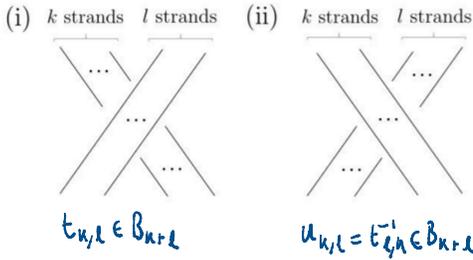
BUT generally not for n-particle system ($n > k$).

↳ **Consequence**: Can observe superpositions over $\{V_X^{i_1, \dots, i_n}\}$ using observables on larger system.

Fusion and the rest

Theorem

Fusion and the rest



Theorem

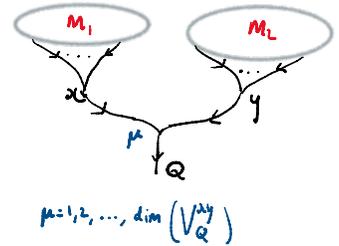
$$|\psi_2\rangle = P_{m_1, m_2}(t_{k,l}) |\psi_1\rangle = e^{i(u_Q - u_x - u_y)} |\psi_1\rangle$$

- In the theorem, m_1 is collection of k particles in sector x
- " , m_2 " l " y
- " , $m_1 m_2$ " $k+l$ " Q

- INTERPRETATION : - m_1 & m_2 behave as quasiparticles under exchange
- their superselection sectors determine their exchange phase $\{e^{i(u_Q - u_x - u_y)}\}_{x,y}$
- Superselection sector = "type" of quasiparticle = "topological charge"
- We say collection m_1 of quasiparticles fuse to a quasiparticle of charge x
- " m_2 " fuse " y
- x & y must fuse to Q (cbe $\dim(V_Q^{xy}) = 0$)
- Total fusion outcome / charge of system is fixed (superselection)

$$V_Q^{m_1 m_2} = V_Q^{m_1 m_2} = \bigoplus_{x,y} V_x^{m_1} \otimes V_Q^{xy} \otimes V_y^{m_2}$$

"FUSION (HILBERT) SPACES"



$$P_{m_1, m_2}(\beta_\mu) |\psi\rangle = e^{i\mu\alpha} |\psi\rangle$$

$$P_{m_1, m_2}(\alpha) |\psi\rangle = e^{i\mu\beta} |\psi\rangle$$

$$P_{m_1, m_2}(\beta_\mu) |\psi\rangle = e^{i\mu\alpha} |\psi\rangle$$

Corollary
Fusion is commutative and associative.

Proof : Commutativity : Eigenspaces of β_k do not depend on order of quasiparticles

Associativity : Given e.g. above, we can recursively partition m_1 in any way we want etc.

The root label (indexing eigenspaces of β_{m_1}) is invariant.

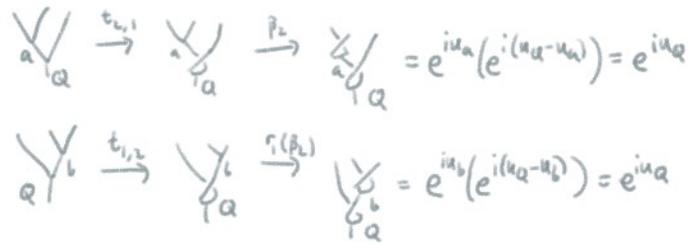
(□)

Theorem (Superselection braid by recursion)

Let $n \geq 2$. For any $k, l \in \mathbb{Z}_+$ s.t. $k+l=n$, β_n is given by

- (i) $[\beta_l \cdot r_l(\beta_n)] t_{k,l}$
 - (ii) $t_{k,l} [\beta_n \cdot r_n(\beta_l)]$
 - (iii) $\beta_l \cdot t_{k,l} \cdot \beta_n$
 - (iv) $r_l(\beta_n) \cdot t_{k,l} \cdot r_n(\beta_l)$
- $r_a: B_n \rightarrow B_{n+a}$
 $\sigma_i \mapsto \sigma_{i+a}$

- Monoidal structure encoded by symmetries of β_n



- Given any fusion tree, β_n exchanges incoming branches at every fusion vertex. Destroys all statistical phases associated to internal nodes!

Vague image: Stokes theorem.

$\hookrightarrow \beta_n$ destroys all internal info and only leaves "boundary info"



Note How close are we to unitary ribbon fusion categories (URFCs)?

\hookrightarrow For any topological charge g , demand existence of charge \bar{g} s.t. $V^{i\bar{i}} = \bigoplus_Q V_Q^{i\bar{i}}$ has $\dim(V_Q^{i\bar{i}}) \neq 0$ for $Q = \mathbb{1}$ (+ some constraint on associator for triple (g, \bar{g}, g))

"KITAEV'S DUALITY AXIOM"

$\mathbb{1} \sim$ "vacuum" $\begin{matrix} i \\ \diagdown \\ \mathbb{1} \\ \diagup \\ \bar{i} \end{matrix}$ (Corollary (Kitaev) $\dim(V_{\mathbb{1}}^{g\bar{g}}) = \dim(V_{\mathbb{1}}^{g\bar{g}}) = \delta_{g\bar{g}}$)

\hookrightarrow Write $N_c^{ab} := \dim(V_c^{ab})$ "fusion coefficients"

\hookrightarrow Altogether, we have an algebraic structure corresponding to a URFC \mathcal{C} !

"fusion algebra" $\left\{ \begin{array}{l} - |\text{Irr}(\mathcal{C})| = \{ \text{superselection sectors a.k.a. topo. charges} \}$ is finite ($\because \rho^{g, i, \bar{i}}: \beta_g \rightarrow U(\mathcal{H})$ finite and has duals) \\ - $N_c^{ab} < \infty \quad \forall a, b, c$ \end{array} \right.

- \mathcal{C} is braided, monoidal, semisimple and \mathbb{k} -linear. Skeletal data is unitary.

"fusion commutative/associative" \leftrightarrow " \otimes "

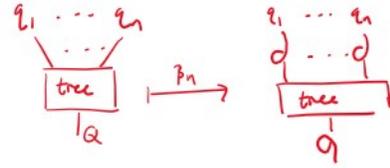
- FACT (ENO): Unitary braided fusion category admits unique unitary ribbon structure

$$\sigma_1^g = \sum_{g \in \text{Irr}(\mathcal{C})} \sigma_g \psi_g$$

Theorem

Given n -quasiparticle system $V^{i_1 \dots i_n}$, the spectrum of β_n^2 is given by $\left\{ \frac{\sigma_g}{\sigma_{i_1} \dots \sigma_{i_n}} \right\}_Q$ where Q indexes the simple objects in $g_1 \otimes \dots \otimes g_n$

↳ The superselection braid also encoded the ribbon structure!



Conclusion & Outlook

- We developed a precise notion of exchange symmetry for spatially localized quasiparticles in (2+1)D using a functor \mathcal{Z} : "coloured braid groupoid" \rightarrow FdHilb
- We recovered URFCs as an algebraic framework for anyon using exchange symmetry in (2+1)D and the following prescription of **postulates**:

A1. Two-dimensional quasiparticles are <i>spatially localised</i> phenomena.	← Condensed matter theory
A2. (i) The Hilbert space of finitely many quasiparticles is finite-dimensional. (ii) A theory of anyons has finitely many distinct topological charges.	← "finiteness postulate"
A3. For any topological charge q , there exists a dual charge \bar{q} such that a certain associativity condition is satisfied with respect to their fusion.	← "Kitaev's duality axiom"

- A1 & A2(i) sufficient for recovering core braiding / fusion properties of anyons
- A2(ii) & A3 let us make contact w/ URFCs

• In Shi et al., the authors also try to derive anyonic structure from some underlying physical principles

	Our approach	Approach of Shi et al.
Physical principle	Exchange symmetry	Entanglement area law
Construction	Local reps of coloured braid groupoid	Information convex sets

Conjectured law:
 $S(A) = \alpha L - \gamma$ (*)

- ↳ Shi et al. recover fusion rules for anyons. In particular, their work:
 - (1) Obviates A3: they show dual is unique and annihilation is unique
 - (2) shows $A2(i) \Rightarrow A2(ii)$
 - (3) Does not make contact with F/R-symbols
- ↳ Combining both approaches, we get a nice-looking set of postulates:

P1. Two-dimensional quasiparticles are <i>spatially localised</i> phenomena.
P2. The Hilbert space of finitely many quasiparticles is finite-dimensional.
P3. The system of quasiparticles satisfies entanglement area law formula (*)

Future Directions

(1) Higher-dim TQFTs. E.g. simplest case is loops in \mathbb{R}^{3+1}



"Motion group" = LB_n (loop braid group)

Can we use exchange symmetry to gain insights into algebraic structure of loop-excitations?

"loop superselection braid?" ...

- (2) A puzzle :
- Given a fusion rule $g \otimes g = \mathbb{1} \oplus \alpha \oplus \bar{\alpha} \oplus \dots$, we know $R_{\alpha}^{2g} = R_{\bar{\alpha}}^{2g}$
 - Our construction thus precludes such fusion rules (since distinct simples in $g \otimes g$ label distinct eigenspaces)
 - $su(3)_3$ contains such a fusion rule and corresponds to a physically realizable model

$\hookrightarrow \Rightarrow su(3)_3$ does not have superselection sectors realized by exchange operator!

(- Guess : can we use theory of anyon condensation to move between our construction and theories beyond our construction)

"non-categorifiable fusion algebra" $\xrightarrow{\text{condensation}}$ $UR=C$

- Could we use this to learn more about microscopic models for theories of anyons?