Remarks on the bicategory of von Neumann algebras

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Ohio State University OSU

March 12, 2021

Plan for the talk

- Bicategories
  - Review
  - Bicategories of C^*-algebras
  - The Buss-Meyer-Zhu programme

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  - First open questions on W^*
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- Double categories of von Neumann algebras
  - Review on double categories
  - The double category BDH
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Bicategories

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All composition operations are assumed to be strictly associative and strictly unital. Vertical and horizontal compositions are assumed to be compatible, i.e. satisfy the Echange property.

Diagrammatic representation: Globular diagrams

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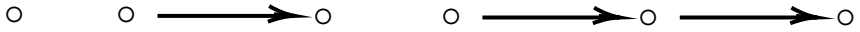

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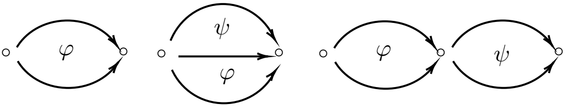

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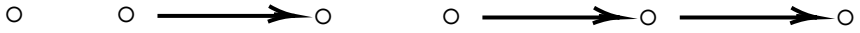

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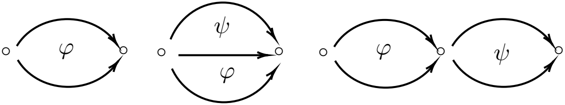
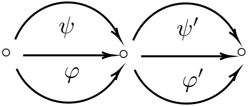


Exchange relation:



Diagrammatic representation: Shaded wire diagrams

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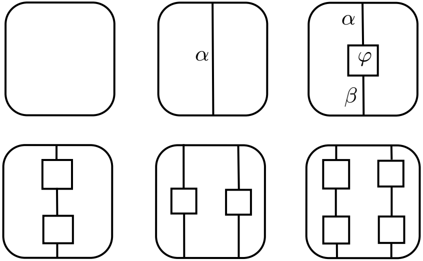

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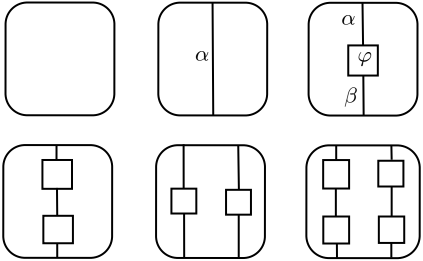

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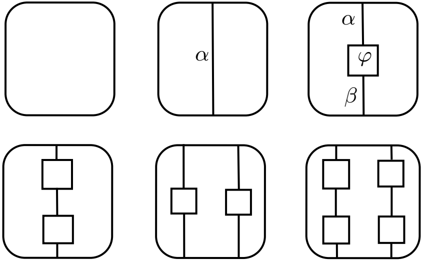

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Notation: We will also represent 2-morphisms as solid arrows, i.e. $\varphi : \alpha \Rightarrow \beta$ will denote a 2-morphism from α to β.

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- **Associator:** A vertically invertible 2-morphism

$$A_{\alpha,\beta,\gamma} : (\alpha * \beta) * \gamma \Rightarrow \alpha * (\beta * \gamma)$$

  for every triple (α, β, γ) of compatible horizontal morphisms in B, natural with respect to (α, β, γ).

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Satisfying the MacLane pentagon and triangle equations, i.e. satisfying:

MacLane equations

$$\begin{tikzcd}
 & & (\alpha\beta)(\gamma\eta) \arrow[dll, dash, "A"'] \arrow[drr, dash, "A"] & & \\
\alpha((\beta\gamma)\eta) & & & & ((\alpha\beta)\gamma)\eta \arrow[dl, dash, "A"] \\
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Basic examples

- **Locally discrete 2-categories:** Let C be a category. $\underline{C}$ denotes the 2-category whose 0- and 1-morphisms are objects and morphisms of C and whose only 2-morphisms are identities. Horizontal 1-dimensional composition is defined by the composition operation of C. We call $\underline{C}$ the locally discrete 2-category associated to C.

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Bicategories of C^*-algebras

1. ***-representations:** $\mathfrak{C}^*(2)$ denotes the 2-category with C^*-algebras as objects, *-representations, i.e. non-degenerate *-morphisms $f : A \to \mathcal{M}(B)$ (eq. strongly continuous $f : \mathcal{M}(A) \to \mathcal{M}(B)$) as horizontal morphisms and unitary intertwiners as 2-morphisms. Horizontal 1-dim composition is the usual composition of morphisms, vertical 2-dim composition is product of intertwiners and 2-dim horizontal composition is: $f, g : A \rightrightarrows B$, $f', g' : B \rightrightarrows C$, $u : f \Rightarrow g$ and $v : f' \Rightarrow g'$ then $u * v = vf'(u) = g'(u)v$. Strict 2-category.

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Weak isomorphisms

Lemma

Let A, B C^*-algebras. Let $f : A \to \mathcal{M}(B)$ be a *-representation of A in B. f is a weak isomorphism in $\mathfrak{C}^*(2)$ if and only if f restricts to a C^*-algebra isomorphism from A to B. Thus A, B are weakly isomorphic in $\mathfrak{C}^*(2)$ if and only if A, B are isomorphic.

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Landsman N. P. Bicategories of operator algebras and Poisson manifolds. Fields Institute communications, 2001, Vol. 30, 271-286

Weak group actions

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Satisfying:

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& id_a * \alpha_g \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha_g * \alpha_1 \arrow[rr, dash, "\omega(g,1)"] \arrow[dr, dash, swap, "\alpha_g * u"] & & \alpha_{g1} = \alpha_g \\
& \alpha_g * id_a \arrow[ur, dash] &
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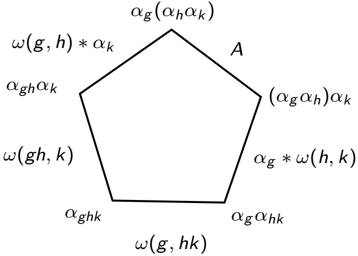
$$

and:

Pentagon axiom



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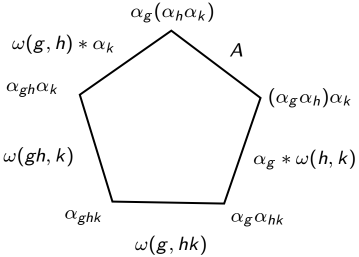

Intuition: The above cocycle equations tell us how to substitute expected composite values of elements of G under the action α. The 2-morphisms u and ω are part of the data provided by α.

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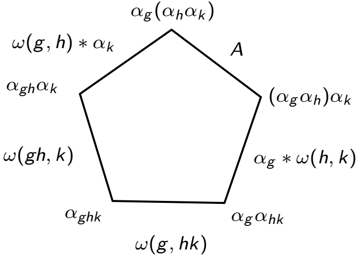

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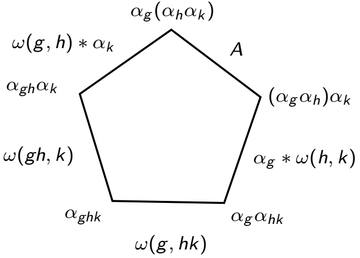

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Let A be a C^*-algebras. Let G be a group. Weak actions of G on A in $\mathfrak{C}^*(2)$ are the Busby-Smith twisted actions of G on A [Busby, Smith 70'].

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Conjecture: Weak actions of inverse semigroups on $\mathfrak{Corr}(2)$.

The bicategory of von Neumann algebras

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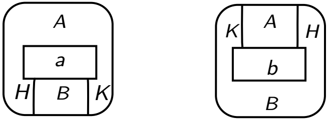

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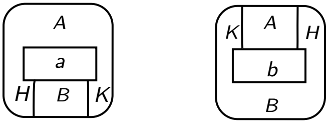

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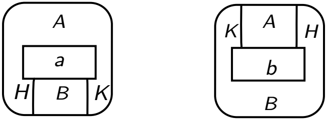

Strong Morita equivalence carries some formal homotopy information. We have a pictorial calculus telling us when two vN algebras are strong Morita equivalent.

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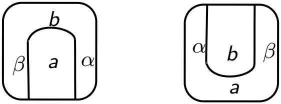

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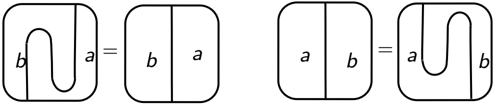

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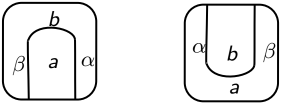

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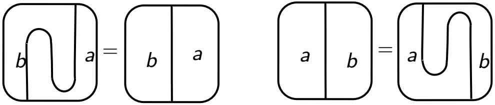

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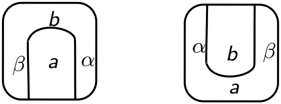

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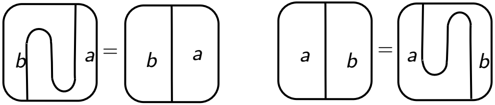

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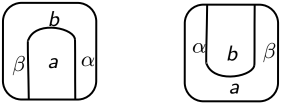

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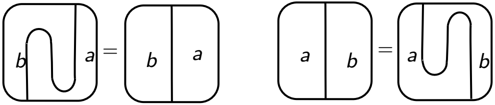

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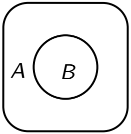

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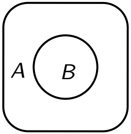

Bartels A., Douglas C.L., Hénriques A. Dualizability and index of subfactors. Quantum Topology 5 (2014), 289-345.

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Symmetric monoidal bicategories

Symmetric monoidal bicategories are rather technical objects. Their definition requires coherence data to be deined in terms of vertically invertible 2-morphisms satisfying the Zamolodchikov tetrahedral equations. See:

M. M. Kapranov, V. A. Voevodski, 2-categories and Zamolodchikov tetrahedra equations. Quantum and infinite dimensional methods, 2; 177-260; 1994

Alternative: Lift W^* to a symmetric monoidal double category. Shulman M. A., Constructing symmetric monoidal bicategories. arXiv:1004.0993 Then lift this structure to a bicategory internal to symmetric monoidal categories Douglas C. L., Hénriques A. Internal bicategories. arXiv:1206.4284. Problem already considered in constructing a symmetric monoidal tricategory of coordinate free conformal nets. **Key idea:** lift to a double category, then define monoidal structure. Also classically done for **Mod**.

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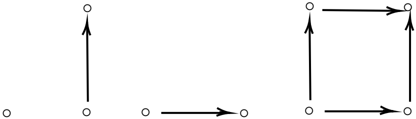

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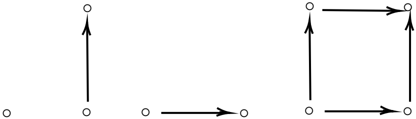

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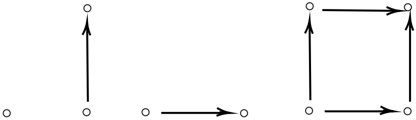

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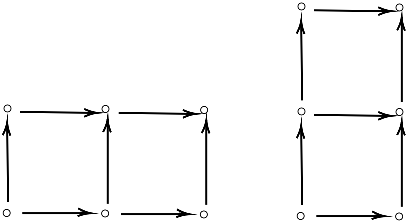

The horizontal bicategory

Let C be a double category. A square in C is globular if it is of the form:

$$\begin{tikzcd} \circ \arrow[r] \arrow[d, equal] & \circ \arrow[d, equal] \\ \circ \arrow[r] & \circ \end{tikzcd}$$

Objects, horizontal morphisms and globular squares of C form a bicategory, denoted by HC and called the horizontal bicategory of C. The function $C \mapsto HC$ extends to a functor $H : \mathbf{dCat} \to \mathbf{bCat}$.

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$$\begin{tikzcd} \circ \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\varphi"] & \circ \arrow[d, equal] \\ \circ \arrow[r] & \circ \end{tikzcd}$$

where φ is a 2-morphism in B. $\mathbb{H}B$ referred to as the trivial double category associated to B. The function $B \mapsto \mathbb{H}B$ extends to an embedding $\mathbb{H} : \mathbf{bCat} \to \mathbf{dCat}$. H and $\mathbb{H}$ are related via $\mathbb{H} \dashv H$.

Another right inverse to H

Let B be a 2-category. Write $\mathbf{Q}B$ for the double category whose squares are of the form:

$$\begin{tikzcd} \circ \arrow[r, "\beta"] & \circ \\ \circ \arrow[u, "\gamma"] \arrow[r, "\alpha"'] \arrow[ur, phantom, "\varphi"] & \circ \arrow[u, "\eta"'] \end{tikzcd}$$

where φ is a 2-morphism, in B, from $\eta\alpha$ to $\beta\gamma$. We denote any such square by a quintet $(\varphi; \alpha, \gamma, \beta, \eta)$ and we call $\mathbf{Q}B$ the Ehresmann double category of quintets of B.

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Think of $\mathbb{H}$ and $\mathbf{Q}$ as ways of lifting a bicategory to a double category.

Main difference between $\mathbf{Q}$ and $\mathbb{H}$: The category of objects of the corresponding double category.

The double category of algebras

Write [**Mod**] for the double category whose squares are of the form:

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\begin{tikzcd}
C \arrow[r, "N"] \arrow[dr, phantom, "\varphi"] & D \\
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where A, B, C and D are algebras, ${}_AM_B$ and ${}_CN_D$ are bimodules, $f : A \to C$ and $g : B \to D$ are unital algebra morphisms, and $\varphi : M \to N$ is a linear transformation such that the equation:

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with A, B, C, D von Neumann algebras, ${}_A\mathcal{H}_B$ and ${}_C\mathcal{K}_D$ bimodules, $f : A \to C$, $g : B \to D$ *-morphisms and $T : H \to K$ bounded s.t:

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BDH identity and composition

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Theorem (Bartels, Douglas, Henriques '14)

There exist functors

$$L^2 : Fact^{<\infty} \to Mod_1^{<\infty}$$

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Technique: Use of the theory of minimal conditional expectations for finite index subfactors [Kosaki 91'] in an essential way. No version of these techniques for infinite index avialable!

The double category BDH

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• *BDH* directly recognizes strong Morita equivalence, finite index, isomorphisms of factors.

Open questions

Open question: Is there a double category of general von Neumann algebras (not-necessarily factors) and von Neumann algebra morphisms C such that $HC = W^*$ and such that BDH is a sub-double category of C? The theory of von Neumann algebras does not give us direct tools to extend BDH to general morphisms.

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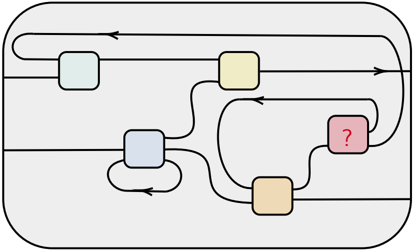

Solutions

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[O'20] There exists a double category of factors and general morphisms $\tilde{Q}_{Fact}$ such that $H\tilde{Q}_{Fact} = Fact$ having BDH as sub-double category. Q_{Fact} and $\tilde{Q}_{Fact}$ are related via a non-trivial double projection and are not double-equivalent.

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[O'20] If we write W^*_{epi} for the category of general von Neumann algebras and epimorphisms, then there exists a double category C^Φ satisfying $HC^\Phi = W^*_{epi}$. C^Φ is constructed using a version of the Grothendieck construction for an **End**-indexing Φ of W^*_{epi}.

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Thank you

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