An overview of the Baum-Connes conjecture

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Introduction

The Kadison-Kaplansky conjecture

Remark : $\mathbb{C}[\Gamma]$ replace by $C^*_r(\Gamma) := \overline{\lambda(\mathbb{C}[\Gamma])}$

↳ left regular rep.

- Let Γ be a (countable) discrete group.
- How to construct idempotents in $\mathbb{C}[\Gamma]$?
  - Assume that Γ has non-trivial torsion.
    - Let $\gamma \in \Gamma$ of order $n > 1$.
    - If ω is a nth root of unity, then the element

$$\boxed{p_\omega := \frac{1}{n}\sum_{i=0}^{n-1}\omega^i\gamma^i} \qquad p_\omega^2 = p_\omega$$

is an idempotent in $\mathbb{C}[\Gamma]$.
  - Assume that Γ is torsion-free.
    - We don't know how to construct idempotents $\neq 0, 1$.

Kadison-Kaplansky conjecture (1949)

If Γ is torsion-free, then $C_r^*(\Gamma)$ has no nontrivial idempotents.

- If Γ is abelian, then we have (Pontryagin duality)

$\widehat{\Gamma}$, compact discret.

Fourier Transfor.

$$\boxed{C_r^*(\Gamma) \cong C(\widehat{\Gamma}).}$$

- Moreover, the following are equivalent:

Th. (Pontryagin)

1. $C_r^*(\Gamma)$ has no nontrivial idempotents,
2. $\widehat{\Gamma}$ is connected,
3. Γ is torsion-free.

Algebraic question of the begining turns into a topological condition.

K-theory

K_0 : built from projections.

- Let A be a (unital) C^*-algebra. K_1 : built from unitaries.
- We will look at two kind of elements: self-adjoint idempotents and unitaries.
  - Given $p, q \in \mathcal{P}_\infty(A)$ $:= \bigcup_n \mathcal{P}(M_n(A))$, we put

$$\underbrace{p}_{\in M_n} \underbrace{\sim_0}_{\text{Murray-von Neumann.}} \underbrace{q}_{\in M_m} \Leftrightarrow \exists v \in \underbrace{\mathcal{M}_\bullet(A)}_{n \times m} \text{ such that } \boxed{p = v^* v \text{ and } q = vv^*}$$

  - Given $u, v \in \mathcal{U}_\infty(A)$, we put $:= \bigcup_n \mathcal{U}(M_n(A)) = \varinjlim \mathcal{U}(M_n(A))$

$$u \sim_1 v \Leftrightarrow \exists k \geqslant \max\{n_u, n_v\} \text{ such that } u \oplus 1_{k-n_u} \sim_h v \oplus 1_{k-n_v}$$

- Put $K_0(A) := \boxed{Gr\left(\mathcal{P}_\infty(A)/\sim_0\right)}$ and $K_1(A) := \mathcal{U}_\infty(A)/\sim_1$.

$\{[p]_0 - [q]_0\}$.

$= \mathcal{U}_\infty(A)/\mathcal{U}_\infty(A)_0$

Examples

Remark: K_0 says p and q have the same range ($\Leftrightarrow$ same trace).

$\forall n \in \mathbb{N}$.

- If $A := \mathcal{M}_n(\mathbb{C})$, then

$$K_0(\mathcal{M}_n(\mathbb{C})) \cong \mathbb{Z} \text{ and } K_1(\mathcal{M}_n(\mathbb{C})) \cong (0)$$

$Tr.\ \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathbb{Z}$
↳ additive
↳ $0 \longmapsto 0$.
↳ Tr preserves $\sim_0$

universal property of K_0

Fact: $\mathcal{U}(\mathcal{M}_n(\mathbb{C}))$ is connected.

$K_0[Tr] : K_0(\mathcal{M}_n(\mathbb{C})) \longrightarrow \mathbb{Z}$
↳ it's isom.

$$\Longrightarrow \quad \mathbb{C}[\Gamma] \cong C^*_r(\Gamma) \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$$

where $r = \#$ conjugacy classes of Γ.

- If Γ is a finite group, then

$$\boxed{K_0(\mathbb{C}[\Gamma]) \cong R(\Gamma) \text{ and } K_1(\mathbb{C}[\Gamma]) \cong (0)}$$

K_0 and K_1 are additive.

Ingredients: (1) suspension: $SA := A \otimes C_0(\mathbb{R})$.

(2) Bott periodicity: $K_0(A) \cong K_1(SA)$

(3) $K_*(\tilde{A}) \cong K_*(A) \oplus \mathbb{Z}$

- If $A := C(S^1)$, then

$$K_0(C(S^1)) \cong \mathbb{Z} \text{ and } K_1(C(S^1)) \cong \mathbb{Z}$$

- $S\mathbb{C} = C_0(\mathbb{R})$

$\Rightarrow$ $K_0(\mathbb{C}) \cong K_1(C_0(\mathbb{R})) \Rightarrow K_1(C_0(\mathbb{R})) \cong \mathbb{Z}$

$K_1(\mathbb{C}) \cong K_0(C_0(\mathbb{R})) \Rightarrow K_0(C_0(\mathbb{R})) \cong (0)$.

- $S^1 :=$ 1-point compact. of $\mathbb{R} \Rightarrow C(S^1) \cong \widetilde{C_0(\mathbb{R})}$

$\Rightarrow K_*(C(S^1)) \cong K_*(C_0(\mathbb{R})) \oplus \mathbb{Z}$

K-homology : Atiyah in the operator algebraic framework.

- X, locally compact space with $\Gamma \curvearrowright X$ by homeomorphisms $\Rightarrow$ $\Gamma \curvearrowright C_0(X)$: $\gamma \cdot f(x) := f(\gamma^{-1}x)$, $\forall \gamma \in \Gamma$, $x \in X$, $f \in C_0(X)$.

Generalized elliptic Γ-operator

A generalized elliptic Γ-operator on X is a triple $((H,u),\pi,F)$ where

Γ-cycle

- u is a unitary representation of Γ on the Hilbert space H,
- $\pi : C_0(X) \longrightarrow \mathcal{B}(H)$ is a Γ-equivariant $*$-homomorphism
- and $F \in \mathcal{B}(H)$ self-adjoint

$\rightsquigarrow$ notion of homotopy

such that

$$(*)\quad \pi(f)(F^2 - id),\ [\pi(f),F],\ \pi(f)(\gamma \cdot F - F) \in \mathcal{K}(H),$$

for all $f \in C_0(X)$ and all $\gamma \in \Gamma$.

K-homology

- $K_1^\Gamma(X) := K^1_\Gamma(C_0(X)) := \dfrac{\text{gener. elliptic } \Gamma\text{-operators}}{\text{" homotopy "}}$.
- $K_0^\Gamma(X) := K^0_\Gamma(C_0(X)) := \dfrac{\textit{even} \text{ gener. elliptic } \Gamma\text{-operators}}{\text{homotopy}}$.

↳ $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$

↳ $u = \begin{pmatrix} u_0 & 0 \\ 0 & u_1 \end{pmatrix}$

↳ $F = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}$ (F self-adjoint)

Examples

$\Longrightarrow C_0(X) = \mathbb{C}$

- If Γ is finite and $X := \{*\}$, then
  - $K_0^\Gamma(\{*\}) \cong R(\Gamma)$.
  - $K_1^\Gamma(\{*\}) \cong (0)$.

$R(\Gamma) \longrightarrow K_0^\Gamma(\{*\})$

$[u_0] - [u_1] \longmapsto \left[\begin{pmatrix} u_0 & 0 \\ 0 & u_1 \end{pmatrix}, \Gamma, 0 \right]$

scalar multip.

spectral properties of F

Examples

- If $\Gamma := \mathbb{Z}$, $X := \mathbb{R}$ and $\mathbb{Z} \curvearrowright \mathbb{R}$ by translation, then
  - $H := L^2(\mathbb{R})$ with representation u of $\mathbb{Z}$ given by

    $$\to \quad u_n(f)(t) := f(t-n) \; \forall n \in \mathbb{Z}, f \in L^2(\mathbb{R}), t \in \mathbb{R}.$$

  - $\pi : C_0(\mathbb{R}) \longrightarrow \mathcal{B}(L^2(\mathbb{R}))$ defined by (pointwise) multiplication.
  - technical $F : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ given by multiplication by $\frac{x}{\sqrt{1+x^2}}$.
  - $\underbrace{[((L^2(\mathbb{R}), u), \pi, F)]}_{\hookrightarrow \text{ generates}} \in K_1^{\mathbb{Z}}(\mathbb{R}) \cong \mathbb{Z}$, $K_0(\mathbb{R}) \cong \mathbb{Z}$.

Kasparov KK-theory

- Generalisation of both K-theory and K-homology.
- Bifunctor $KK(\cdot,\cdot)$ defined on (separable) C^*-algebras.
  - $KK(A,B)$ is an abelian group. : generalized $*$-hom. between A, B.
  - $KK(\mathbb{C},B) = K_0(B)$. $\rightsquigarrow$ K-theory
  - $KK(A,\mathbb{C}) = K^0(A)$. $\rightsquigarrow$ K-homology.
  - *Kasparov product*: → composition law.

    $\underset{C}{\otimes} : KK(A,C) \times KK(C,B) \longrightarrow KK(A,B).$

    ...
- Γ-equivariant version: $KK^\Gamma(A,B) \rightsquigarrow$ *descent principle*:

$$j_\Gamma : KK^\Gamma(A,B) \longrightarrow KK(A \underset{r}{\rtimes} \Gamma, B \underset{r}{\rtimes} \Gamma)$$

The BC assembly map

General ideas

- **Problems.**
  - It is hard to extract structural information of $C_r^*(\Gamma)$.
  - It is hard to compute $K_*(C_r^*(\Gamma))$.
  - Torsion of Γ should be handled in some way.
- **Strategy.**
  - Apply topological/geometrical techniques to study $C_r^*(\Gamma)$.
  - Find a topological space associated to Γ able to encode all relevant information of $K_*(C_r^*(\Gamma))$.
  - Assembly local data (coming from the torsion phenomena) to understand the global space.

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- **Problems**.
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analytical.

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A model for $\underline{E}\Gamma$

$\to$ X is locally of the form $\Gamma \times_{\Lambda} Y$ = $\Gamma \times Y /$ diagonal action.

- X, Hausdorff space.
- $\Gamma \curvearrowright X$ by homeomorphisms. $\longrightarrow$ is induced from a finite subgroup.

Proper action

We say that X is a proper Γ-space or that the action $\Gamma \curvearrowright X$ is proper if for every $x \in X$ there exists a triple (U_x, Λ_x, ρ_x) where

- U_x is a Γ-invariant open neighborhood of x,
- Λ_x is a finite subgroup of Γ
- and $\rho_x : U_x \longrightarrow \Gamma/\Lambda_x$ is a Γ-invariant map.

A model for $\underline{E}\Gamma$

- If Γ is finite $\Rightarrow$ every action of Γ is proper $\Rightarrow$ $\underline{E}\Gamma := \{*\}$
- If Γ is torsion-free $\Rightarrow$ we recover the notion of principal bundle $\Rightarrow$ $\underline{E}\Gamma \overset{\text{def}}{=} E\Gamma$

slide before

Universal example

A universal example for proper actions of Γ, denoted by $\underline{E}\Gamma$, is a proper Γ-space such that if X is another proper Γ-space, then there exists a Γ-map $X \longrightarrow \underline{E}\Gamma$, and any two Γ-maps from X to $\underline{E}\Gamma$ are Γ-homotopic.

universal principal bundle of the classifying space of Γ.

- $\underline{E}\Gamma$ is unique up to Γ-homotopy.

(*) $\underline{E}\Gamma$ always exists.

A model for $\underline{E}\Gamma$

Probability measures with finite support

$\underline{E}\Gamma := \{f : \Gamma \longrightarrow [0,1] \mid f$ has finite support and $\sum_{x\in\Gamma} f(x) = 1\}$

- $\underline{E}\Gamma$ is a metric space with the sup-norm:

$$||f-g||_\infty := \sup_{x\in\Gamma}\{|f(x)-g(x)|\}, \ \forall f,g \in \underline{E}\Gamma.$$

- $\Gamma \curvearrowright \underline{E}\Gamma$ properly via

$$\gamma \cdot f(x) := f(\gamma^{-1}x), \ \forall x,\gamma \in \Gamma, \ f \in \underline{E}\Gamma.$$

- $\underline{E}\Gamma$ is a universal example for proper actions of Γ.

Construction of the assembly map

want understand.

- Right hand side: $K_*(C_r^*(\Gamma))$.
- Left hand side: $K_*^{top}(\Gamma) = \boxed{\varinjlim_{\substack{X \subset \underline{E}\Gamma \\ \Gamma-compact}} K_*^{\Gamma}(X)}$: Γ-equiv. K-homology with compact support.
- We can construct a group homomorphism

$$\boxed{\mu^{\Gamma} : K_*^{top}(\Gamma) \longrightarrow K_*(C_r^*(\Gamma)).}$$

Assembly map.

Th.
- *Baum-Connes-Higson's approach* $\rightsquigarrow$ deal directly with K-homology.
- *Kasparov's approach* $\rightsquigarrow$ use KK-theory $+$ descent principle.

Baum-Connes conjecture (1982)

We say that Γ satisfies the Baum-Connes conjecture if the assembly map

$$\mu^{\Gamma} : K_*^{top}(\Gamma) \longrightarrow K_*(C_r^*(\Gamma))$$

is an isomorphism.

Some consequences

1. If the BC conjecture is true, then we would have a method to compute $K_*(C_r^*(\Gamma))$: through topological/geometrical techniques.
2. μ^Γ surjective $\Rightarrow$ Kadison-Kaplansky conjecture for Γ holds.
3. μ^Γ injective $\Rightarrow$ consequences in geometry (*existence of specific invariants on manifolds* $\rightsquigarrow$ *Novikov conjecture*).

Perspectives of the conjecture

Elementary examples

- If $\Gamma := e$, then $C_r^*(\{e\}) = \mathbb{C}$ and $\underline{E}\{e\} = \{*\}$.
- If Γ is finite, then $C_r^*(\Gamma) \cong \bigoplus_{i=1}^{r} \mathcal{M}_{k_i}(\mathbb{C})$ and $\underline{E}\Gamma = \{*\}$.

abstractly isom. ||

  - $K_0(C_r^*(\Gamma)) \cong R(\Gamma)$, $K_1(C_r^*(\Gamma)) \cong (0)$.
  - $K_0^\Gamma(\{*\}) \cong R(\Gamma)$, $K_1^\Gamma(\{*\}) \cong (0)$.
  - Moreover, μ^Γ realizes the isomorphism between $K_*^{top}(\{*\})$ and $K_*(C_r^*(\Gamma))$.

- Γ satisfies the Baum-Connes conjecture!

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- Γ satisfies the Baum-Connes conjecture!

Elementary examples

abelian $\to$ Pontryagin   torsion-free.

- If $\Gamma = \mathbb{Z}$, then $C_r^*(\mathbb{Z}) \cong C(S^1)$ and $\underline{E}\mathbb{Z} = E\mathbb{Z} = \mathbb{R}$

abstractly isom.

  - $K_0(C_r^*(\mathbb{Z})) \cong \mathbb{Z}$, $K_1(C_r^*(\mathbb{Z})) \cong \mathbb{Z}$.
  - $K_0^{\mathbb{Z}}(\mathbb{R}) \cong \mathbb{Z}$, $K_1^{\mathbb{Z}}(\mathbb{R}) \cong \mathbb{Z}$.
  - Moreover, $\mu^{\mathbb{Z}}$ realizes the isomorphism between $K_*^{top}(\mathbb{R})$ and $K_*(C(S^1))$.

- $\mathbb{Z}$ satisfies the Baum-Connes conjecture!

Elementary examples

- If $\Gamma = \mathbb{Z}$, then $C_r^*(\mathbb{Z}) \cong C(S^1)$ and $\underline{E}\mathbb{Z} = E\mathbb{Z} = \mathbb{R}$
  - $K_0(C_r^*(\mathbb{Z})) \cong \mathbb{Z}$, $K_1(C_r^*(\mathbb{Z})) \cong \mathbb{Z}$.
  - $K_0^{\mathbb{Z}}(\mathbb{R}) \cong \mathbb{Z}$, $K_1^{\mathbb{Z}}(\mathbb{R}) \cong \mathbb{Z}$.
  - Moreover, $\mu^{\mathbb{Z}}$ realizes the isomorphism between $K_*^{top}(\mathbb{R})$ and $K_*(C(S^1))$.
- $\mathbb{Z}$ satisfies the Baum-Connes conjecture!

How to tackle the conjecture?

- Most of the known proofs of BC are based in this method and it was formalised by J.-L. Tu (1999).
- Put $\mathscr{L}_\Gamma := \{$*proper* $\Gamma - C^* -$ algebras$\}$: $\mathrm{Ind}_\Lambda^\Gamma(B)$, $\Lambda < \Gamma$ finite.
- Assume that there exist

1. $A \in \mathscr{L}_\Gamma$,
2. $\alpha \in KK^\Gamma(A, \mathbb{C})$ (*Dirac element*), $\in$ K-homol.
3. and $\beta \in KK^\Gamma(\mathbb{C}, A)$ (*dual Dirac element*) $\in$ K-theor.

such that $\gamma := \underbrace{\beta \underset{A}{\otimes} \alpha}_{\text{Kasparov}} = \mathbb{1}_\mathbb{C} \in KK^\Gamma(\mathbb{C}, \mathbb{C})$.

Construct a "γ-element" (+ condition).

always !! injectivity of μ^Γ

Then the Baum-Connes conjecture holds for Γ!

Obstructions?

Property [T]: "trivial reps. is an isolated point among unitary reps of Γ".

Theorem (V. Lafforgue, 2012)

If Γ is a hyperbolic group, then Γ satisfies the Baum-Connes conjecture.

- Many hyperbolic groups have property (T): lattices in sympectic groups.
- Very rare to find (infinite) discrete groups with property (T) for which we know to show the BC conjecture...

γ-element $\in KK^{\Gamma}(\mathbb{C}, \mathbb{C})$
$\hookrightarrow$ underlying rep. π

$1_{\mathbb{C}} \in KK^{\Gamma}(\mathbb{C}, \mathbb{C})$
$\hookrightarrow$ underlying rep. = trivial.

Other positive answers

Haagerup property : " ∃ unit. rep. of Γ "containing" the trivial "

Theorem (N. Higson & G. Kasparov, 2001)

If Γ has the Haagerup property, then Γ satisfies the Baum-Connes conjecture.

- Very large class of groups: abelian, finite, amenable, K-amenable...

→ Construct a γ-element and $\gamma = 1$.

Beyond discrete groups

- $\Gamma \rightsquigarrow G$, locally compact group.
- $G \rightsquigarrow \mathcal{G}$, locally compact groupoid.
- *Groups* $\rightsquigarrow X$ metric spaces: coarse geometry.

...

- In all cases, we can include *coefficients*:

$$\mu_A^G : K_*^{top}(G, A) \longrightarrow K_*(A \underset{r}{\rtimes} G)$$

  - The Baum-Connes conjecture with coefficients turns out to be false (*Higson-Lafforgue-Skandalis, 2001*).

Beyond discrete groups

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- *Groups* $\rightsquigarrow X$ metric spaces: coarse geometry.

...

- In all cases, we can include *coefficients*:

$C^*_r(G)$ when $A=\mathbb{C}$

$$\mu_A^G : K_*^{top}(G, A) \longrightarrow K_*(A \rtimes_r G)$$

  - The Baum-Connes conjecture with coefficients turns out to be false (*Higson-Lafforgue-Skandalis, 2001*).

Beyond geometry

So far, the formulation of BC has a fundamental geometrical component...

> Can we obtain a *quantum* Baum-Connes conjecture?
>
> Groups, $G \rightsquigarrow$ Quantum Groups, $\mathbb{G}$

Meyer-Nest's approach

General ideas

- **Problems**.
  - The geometry behind BC avoids its translation into a *quantum* framework.
  - The BC assembly map makes sense only for K-theory $\rightsquigarrow$ define a BC assembly map for other equivariant homology theories?
  - $K_*^{top}(\Gamma)$ creates, sometimes, more problems than $K_*(C_r^*(\Gamma))$.
- **Strategy**.
  - Adopt a *categorical* approach.
  - Find *generating* subcategory: $\mathscr{L}_\Gamma \leftrightsquigarrow$ *torsion of* Γ.
  - Replace $K_*^{top}(\Gamma)$ by other K-theory groups that *approximate* $K_*(C_r^*(\Gamma))$ *in terms of* $\mathscr{L}_\Gamma$ through spectral sequences.

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The Kasparov category

- If Γ is a (countable) discrete group, $\mathscr{K}\mathscr{K}^{\Gamma}$ denotes the Γ-*equivariant Kasparov category*:
  - $Obj(\mathscr{K}\mathscr{K}^{\Gamma}) :=$ separable Γ-C^*-algebras.
  - $Hom_{\mathscr{K}\mathscr{K}^{\Gamma}}(A, B) := KK^{\Gamma}(A, B)$.
- Suspension of C^*-algebras, Σ, is an auto-equivalence (by Bott periodicity).
- Given an equivariant $*$-homomorphism $f : A \longrightarrow B$, a mapping cone triangle is the following diagram

$$\Sigma(B) \longrightarrow C_f \longrightarrow A \xrightarrow{f} B.$$

- **Theorem** (Meyer-Nest, 2006): $(\mathscr{K}\mathscr{K}^{\Gamma}, \Sigma, \Delta_{\Sigma})$ is a triangulated category.

Choice of the complementary pair in $\mathscr{K}\mathscr{K}^{\Gamma}$

- $\mathcal{F} :=$ family of all finite subgroups of Γ.
- *Compactly induced objects*:

$$\mathscr{L}_{\Gamma} := \langle\{A \in Obj(\mathscr{K}\mathscr{K}^{\Gamma}) \mid A \cong Ind_{\Lambda}^{\Gamma}(B),\, \Lambda \in \mathcal{F},\, B \in \mathscr{K}\mathscr{K}^{\Lambda}\}\rangle.$$

- *Compactly contractible objects*:

$$\mathscr{N}_{\Gamma} := \{A \in Obj(\mathscr{K}\mathscr{K}^{\Gamma}) \mid Res_{\Lambda}^{\Gamma}(A) \cong (0)\ \forall \Lambda \in \mathcal{F}\}.$$

- **Theorem** (Meyer-Nest, 2006): $(\mathscr{L}_{\Gamma}, \mathscr{N}_{\Gamma})$ is a complementary pair in $\mathscr{K}\mathscr{K}^{\Gamma}$.

Categorifying the assembly map

- Consider the functor

$$\begin{array}{rccc} F: & \mathscr{K}\mathscr{K}^{\Gamma} & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & A & \longmapsto & F(A) := K_*(A \underset{r}{\rtimes} \Gamma) \end{array}$$

- The *categorical Baum-Connes assembly map* is the natural transformation

$$\eta^{\Gamma} : \mathbb{L}F \longrightarrow F$$

Categorical Baum-Connes conjecture

- We say that Γ satisfies the *(categorical) Baum-Connes conjecture* if η^{Γ} is a natural equivalence.
- We say that Γ satisfies the ***strong*** *(categorical) Baum-Connes conjecture* if $\mathscr{L} = \mathscr{K}\mathscr{K}^{\Gamma}$.

  - *Strong* Baum-Connes conjecture $\leftrightsquigarrow$ Dirac-dual Dirac method.

Reformulation

Reformulation of BC (*R. Meyer & R. Nest, 2006*)

The following assertions are equivalent:

1. Γ satisfies the Baum-Connes conjecture (with coefficients): μ^{Γ}_{A} is an isomorphism, for every Γ-C^*-algebra A.
2. The natural transformation $\eta^{\Gamma}: \mathbb{L}F \longrightarrow F$ is a natural equivalence.

Reformulation II

Reformulation of BC (*R. Meyer & R. Nest, 2006*)

The following assertions are equivalent:

1. Γ satisfies the Baum-Connes conjecture (with coefficients)
2. $F(A) = K(A \rtimes_r \Gamma) = (0)$, for every Γ-C^*-algebra $A \in \mathscr{N}$.
3. If A, B are Γ-C^*-algebras such that $K(A \underset{r}{\rtimes} \Lambda) \cong K(B \underset{r}{\rtimes} \Lambda)$ for every $\Lambda \in \mathcal{F}$, then $K(A \underset{r}{\rtimes} \Gamma) \cong K(B \underset{r}{\rtimes} \Gamma)$.

- **Main problem**: what is torsion for $\mathbb{\Gamma}$ and how to handle it in the previous famework?
  - $\mathbb{\Lambda} \leqslant \mathbb{\Gamma}$ yields torsion for $\mathbb{\Gamma}$.
  - The family of finite discrete quantum subgroups of $\mathbb{\Gamma}$ **does not** cover the whole torsion phenomena for $\mathbb{\Gamma}$!
  - The Induction-Restriction approach must be revisited $\rightsquigarrow$ how to choose the complementary pair in $\mathscr{K}\mathscr{K}^{\mathbb{\Gamma}}$?

Some results

Kadison-Kaplansky.

fails   true

- BC for basic examples: $\widehat{SU_q(2)}$, $\widehat{O^+(n)}$, $\widehat{U^+(n)}$, $\widehat{S_N^+}$ (*C. Voigt and R. Vergnioux, 2011-2015*).
- K-theory computations for their C^*-algebras.
- Stability of BC: it passes through the free product construction (*C. Voigt - R. Vergnioux, 2013*), it passes through the quantum semi-direct product construction (*R. M., 2017*), it passes through the free wreath product construction (*A. Freslon - R. M., 2017*).

recents works about Drinfeld twisted and q-deform.

- Torsion phenomena:
  - Classification results: S_N^+ (*C. Voigt, 2015*), $\mathbb{G} \wr_* S_N^+$ (*A. Freslon - R. M., 2017*), $\Gamma \ltimes \mathbb{G}$ (*P. Fima - R.M., in progress*), projective torsion is given by projective representations (*K. De Commer - R. Nest - R. M., in preparation*).
  - Stability of torsion-freeness: it passes through the free product construction (*Y. Arano - K. De Commer, 2015*), it passes through the quantum semi-direct product construction (*R. M., 2017*), it passes through divisible discrete quantum subgroups (*R. M., 2020*).

- BC formulation:
  - Candidate for $\mathscr{L}_{\mathbb{\Gamma}}$ (*within the works of C. Voigt*).
  - $(\mathscr{L}_{\mathbb{\Gamma}}, \mathscr{N}_{\mathbb{\Gamma}})$ is complementary in $\mathscr{K}\mathscr{K}^{\mathbb{\Gamma}}$ (*Y. Arano - A. Skalski, 2020*).
  - $(\mathscr{L}_{\mathbb{\Gamma}}, \mathscr{N}_{\mathbb{\Gamma}})$ is complementary in $\mathscr{K}\mathscr{K}^{\mathbb{\Gamma}}$ when $\mathbb{\Gamma}$ is <u>permutation torsion-free</u> (*K. De Commer - R. Nest - R. M., in preparation*).

Conclusion

- Understanding the structure of group C^*-algebras.
  - Geometry.
  - Analysis.
  - Representation theory.
  - Category theory.
- Linking mathematics of different flavour and nature.

Thank you!