Symmetries of Operator Algebras

Samuel Evington

University of Münster

Operator Algebras

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Notation

H	Complex Hilbert Space
$B(H)$	Algebra of continuous linear operators $H \to H$
T^*	Adjoint of $T \in B(H)$
	Characterised by $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$

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Uniform Limits

C*-Algebras
e.g. $C(X)$
Non-commutative topology

Pointwise Limits

von Neumann Algebras
e.g. $L^\infty(\Omega)$
Non-commutative measure theory

Formalism for quantum symmetries

- Classical:
  - Automorphisms $\phi : A \to A$.
  - Group theory
  - Actions: homomorphisms $G \to \mathrm{Aut}(A)$.

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  - Tensor categories
  - Actions: tensor functors $\mathcal{C} \to \mathrm{Bim}(A)$.

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Let A be an operator algebra and $\mathcal{C}$ be a tensor category.
Do there exists actions $\mathcal{C} \curvearrowright A$?
To what extent is the action unique?

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- Anomalous Symmetries:
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  - Outer automorphisms $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$.
  - "Actions": homomorphisms $G \to \mathrm{Out}(A)$.
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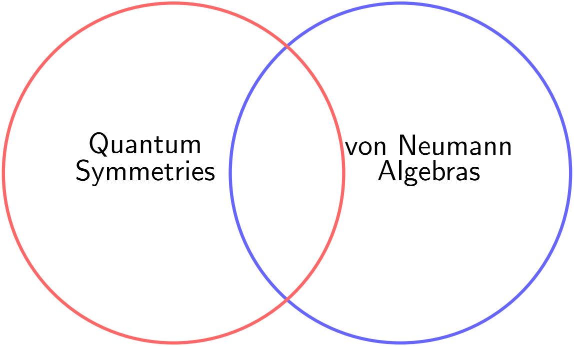
  - Actions $\mathrm{Vec}(G,\omega) \curvearrowright A$ or more precisely actions $\mathrm{Hilb}(G,\omega) \curvearrowright A$.

Big Picture



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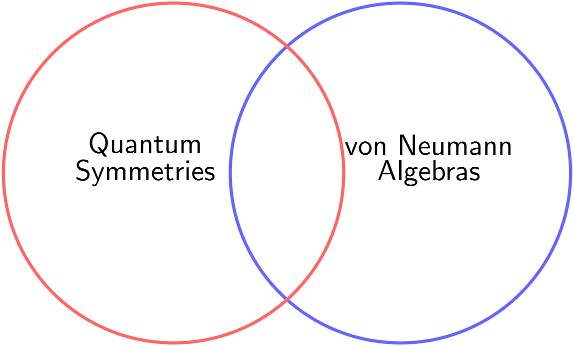

Theorem (Connes)

There is a unique (separably acting) amenable II_1 factor

$$\mathcal{R} = \overline{\bigotimes}_{i\in\mathbb{N}}(\mathbb{M}_2(\mathbb{C}), \mathrm{tr}).$$

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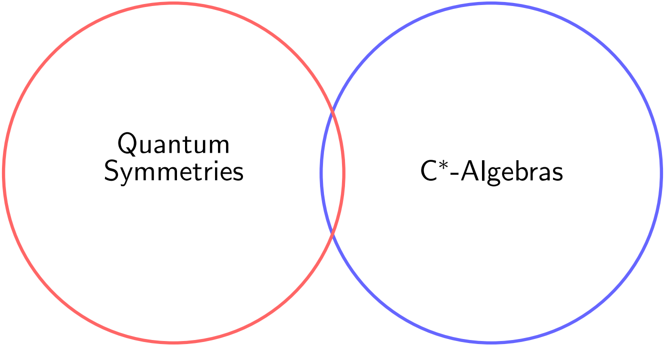

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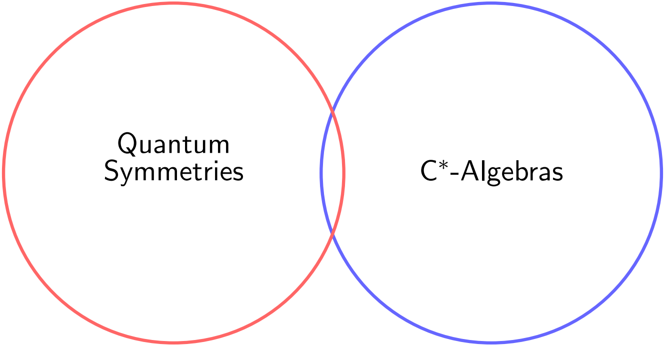
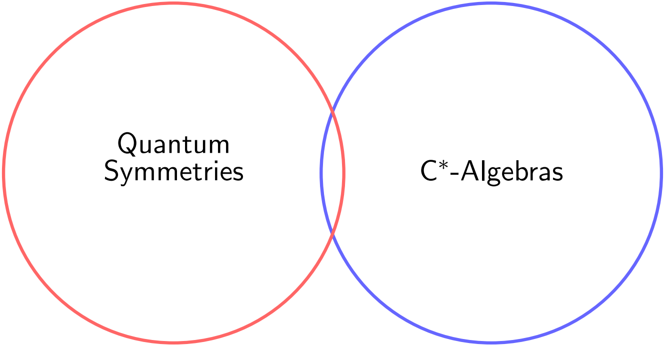


Theorem

There are loads of simple amenable C^*-algebras.

Big Picture



Theorem (2015, The Elliott Programme)

There class of unital, simple, separable, amenable, $\mathcal{Z}$-stable C-Algebras satisfying the UCT is classified by K-theory and traces.*

Symmetries of Operator Algebras

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2. Symmetries of the classifiable C^*-algebras

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- Every $\sigma \in \mathrm{Sym}(\mathbb{N})$ defines an automorphism $\phi_\sigma \in \mathrm{Aut}(\mathcal{R})$ by permuting the tensor factors.

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Theorem

Every countable discrete groups G embeds in $\mathrm{Aut}(\mathcal{R})$.

$\mathrm{Inn}(\mathcal{R})$ and $\mathrm{Out}(\mathcal{R})$

Let u be an unitary in

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- $u \in_\epsilon \overline{\bigotimes}_{i=1}^{N}\mathbb{M}_2(\mathbb{C}) \otimes 1 \otimes 1 \otimes \cdots$ for some N.

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Connes' classification of automorphisms of $\mathcal{R}$

Definition

Let $\phi, \psi \in \mathrm{Aut}(\mathcal{R})$.
Say ϕ and ψ are *conjugate* if $\phi = \theta \circ \psi \circ \theta^{-1}$ for some $\theta \in \mathrm{Aut}(\mathcal{R})$.
Say ϕ and ψ are *outer conjugate* if $\bar{\phi} = \theta \circ \bar{\psi} \circ \theta^{-1}$ for some $\theta \in \mathrm{Out}(\mathcal{R})$.

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- The order $n \in \mathbb{N}$, i.e. the smallest $n \in \mathbb{N}$ such that $\bar{\phi}^n = 1$ in $\mathrm{Out}(\mathcal{R})$.
- When $n < \infty$, an n-th root of unity $\omega \in \mathbb{C}$ such that

$$\phi^n = \mathrm{Ad}(u) \quad \text{and} \quad \phi(u) = \omega u.$$

Where does the n-th root of unity come from?

Theorem (Connes)

Let $\phi \in \mathrm{Aut}(\mathcal{R})$. Suppose $\phi^n = \mathrm{Ad}(u)$. Then $\phi(u) = \omega u$ for some n-th root of unity $\omega \in \mathbb{C}$.

Proof.

We have

$$\begin{aligned} \phi^n \circ \phi &= \phi \circ \phi^n, \\ \mathrm{Ad}(u) \circ \phi &= \phi \circ \mathrm{Ad}(u), \\ \mathrm{Ad}(u) \circ \phi &= \mathrm{Ad}(\phi(u)) \circ \phi, \end{aligned}$$

[Note $\phi(uau^*) = \phi(u)\phi(a)\phi(u)^*$.]

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[Note $\phi(uau^*) = \phi(u)\phi(a)\phi(u)^*$.]
So $\mathrm{Ad}(\phi(u)) = \mathrm{Ad}(u)$.
As $Z(\mathcal{R}) = \mathbb{C}$, this means that $\phi(u) = \omega u$ for some $\omega \in \mathbb{C}$.
Since ϕ^n fixes u, we get that ω is an n-th root of unity.

□

Constructing automorphisms with $\omega \neq 1$

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Theorem (Connes)

View $\mathcal{R} = \overline{\bigotimes}_{i \in \mathbb{N}} \mathbb{M}_n$. Let $\pi_i : \mathbb{M}_n \to \mathcal{R}$ be the embedding into the i-th tensor factor, and let $\theta : \mathcal{R} \to \mathcal{R}$ be the endomorphsim such that $\theta\pi_i = \pi_{i+1}$ for all $i \in \mathbb{N}$.
Let ω be an n-th root of unity. Set

$$u = \sum_{j=1}^{n} \omega^j \pi_1(e_{jj}) \tag{1}$$

$$v = \pi_1(e_{n1})\theta(u) + \sum_{j=1}^{n-1} \pi_1(e_{j,j+1}). \tag{2}$$

Then the sequence $(\mathrm{Ad}(v\theta(v)\theta^2(v)\cdots\theta^k(v)))_{k=1}^{\infty}$ converges pointwise in the weak topology to an automorphism s_n^ω such that $(s_n^\omega)^n = \mathrm{Ad}(u)$ and $s_n^\omega(u) = \omega u$.*

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Since A_θ has a unique trace, ϕ extends to an automorphism of $\mathrm{GNS}_{\mathrm{tr}}(A_\theta)'' \cong \mathcal{R}$.

Generalisations

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  - ω is now an element of $H^3(G, \mathbb{T})$.
- Popa:
  - Actions of amenable tensor categories on $\mathcal{R}$ (via subfactor theory).
  - ω is now the associator.

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C*-algebraic results

Joint work with Sergio Girón Pacheco.

Idea: Replace $\mathcal{R}$ with classifiable C*-algebras

- e.g. UHF algebras, AF algebras, the Jiang–Su algebra $\mathcal{Z}$, ...

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Note: We can also work with tensor functors $\mathrm{Hilb}(G, \omega) \to \mathrm{Bim}(A)$ (with some caveats).

C*-Algebraic Cheat Sheet

C^*-Algebraic Cheat Sheet

- UHF algebras:
  - $\mathrm{UHF}_{\mathfrak{n}} = \bigotimes_{i \in \mathbb{N}} \mathbb{M}_{n_i}(\mathbb{C})$
  - Classified by the *supernatural number* $\mathfrak{n} = n_1 n_2 n_3 \cdots$.
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  - Inductive limits $F_1 \to F_2 \to F_3 \to \cdots$, where F_i finite-dimensional
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- The Jiang–Su Algebra $\mathcal{Z}$:
  - An inductive limit $D_1 \to D_2 \to D_3 \to \cdots$, where $D_i \subseteq C([0,1], \mathbb{M}_{n_i})$
  - No non-trivial projections, $K_0 = \mathbb{Z}$.
  - $K_1 = 0$, unique trace
  - Important because classifiable C*-algebras satisfy $A \otimes \mathcal{Z} \cong A$.

Existence results

By adapting the vN-algebraic constructions, ...

Theorem

For any finite group G and $\omega \in H^3(G; \mathbb{T})$.
There exists a simple AF algebra A with unique trace and a homomorphism $G \to \mathrm{Out}(A)$ with invariant ω.

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In fact, we can take $A = \mathrm{UHF}_{|G|^\infty}$.

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The Jiang–Su algebra $\mathcal{Z}$:

Theorem (E, Girón Pacheco)

For any group G and $\omega \in H^3(G;\mathbb{T})$.
Suppose there exists an embedding $G \to \mathrm{Out}(\mathcal{Z})$ with invariant ω.
Then ω vanishes in $H^3(G;\mathbb{T})$.

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UHF algebras:

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Suppose there exists an embedding $G \to \mathrm{Out}(\mathrm{UHF}_{\mathfrak{n}})$ with invariant ω.
Then the order of ω in $H^3(G;\mathbb{T})$ divides $\mathfrak{n}$ and $|G|$.

The proofs make use of (unitary) algebraic K_1, which has also had a role in Elliott classification programme.

Algebraic K_1

The C*-algebras $\mathcal{R}$, $\mathcal{Z}$ and UHF_s all have $K_1 = 0$, where

$$K_1(A) := \lim_{\rightarrow} \frac{U_n(A)}{\sim_h}.$$

However, unitary algebraic K_1, defined by

$$K_1^{\mathrm{alg}}(A) := \lim_{\rightarrow} \frac{U_n(A)}{DU_n(A)},$$

distinguishes them. We have

$$\begin{aligned} K_1^{\mathrm{alg}}(\mathcal{R}) &= \mathbb{R}/\mathbb{R} = 0 \\ K_1^{\mathrm{alg}}(\mathcal{Z}) &= \mathbb{R}/\mathbb{Z} = \mathbb{T} \\ K_1^{\mathrm{alg}}(\mathrm{UHF}_{\mathfrak{n}}) &= \mathbb{R}/Q(\mathfrak{n}) \end{aligned}$$

Twisted actions of $\mathbb{Z}/n\mathbb{Z}$ on $\mathcal{Z}$

An isomorphism $K_1^{\mathrm{alg}}(\mathcal{Z}) \cong \mathbb{R}/\mathbb{Z}$ is given by

$$[\exp(2\pi i h)] \mapsto \mathrm{tr}(h) + \mathbb{Z}.$$

Consequently,

- the scalar unitaries $\lambda 1_{\mathcal{Z}}$ are a complete set of K_1^{alg} representatives;
- every $\phi \in \mathrm{Aut}(\mathcal{Z})$ preserves K_1^{alg} classes.

Taking K_1^{alg} of the equation

$$\phi(u) = \omega u$$

gives $[u] = [\omega 1_{\mathcal{Z}}] + [u]$. So $[\omega 1_{\mathcal{Z}}]$ is trivial. So $\omega = 1$.

Questions

Any Questions?