Quantum Symmetries of Quantum Metric Spaces

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Outline

Compact Quantum Groups

Compact Quantum Groups: Intuition

Quantum Groups are used as a term to describe certain classes of Hopf algebras.

Theorem (Gelfand Duality)

Every commutative unital C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space X.
Moreover, every unital $*$-homomorphism $C(X) \to C(Y)$ is of the form

$$f \mapsto (y \mapsto f(\phi(y)))$$

for a (unique) continuous map $\phi : Y \to X$.

Gelfand duality $\Rightarrow$ all information about compact Hausdorff spaces is captured by the theory of commutative C^*-algebras.

Compact Quantum Groups: Intuition

Quantum Spaces

Function algebras of compact spaces $\leftrightarrow$ commutative unital C^*-algebras
Function algebras of "quantum spaces" $\leftrightarrow$ non-commutative C^*-algebras

Gelfand duality $\Rightarrow$ structure of a compact *group* G can be phrased entirely in terms of extra structure on the commutative unital C^*-algebra of functions $C(G)$.

Compact Quantum Groups

Compact quantum groups $\leftrightarrow$ consider same extra structure on arbitrary unital C^*-algebras

Compact Quantum Groups: Motivation

G: compact group
$\mathcal{A} = C(G)$ with homomorphisms

- $$\Delta : C(G) \to C(G \times G) \cong C(G) \otimes C(G)$$
  $$\big(\Delta(f)\big)(g, h) = f(g \cdot h)$$

  satisfies $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.
  There is a natural isomorphism
  $$\Psi : C(G) \otimes C(G) \to C(G \times G)$$
  $$f_1 \otimes f_2 \mapsto \Psi(f_1 \otimes f_2)$$
  where $\big(\Psi(f_1 \otimes f_2)\big)(g, h) := f_1(g) f_2(h)$.

Compact Quantum Groups: Motivation

G: compact group
$\mathcal{A} = C(G)$ with homomorphisms

- $$\Delta : C(G) \to C(G \times G)$$
  $$\big(\Delta(f)\big)(g, h) = f(g \cdot h)$$
  satisfies $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.
- $\epsilon : C(G) \to \mathbb{C}$ such that $\epsilon(f) = f(e)$. The unit conditions translate to $(\epsilon \otimes id) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$.
- $S : C(G) \to C(G)$ such that $S(f)(g) = f(g^{-1})$. The inverse conditions translate to
  $m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$ where
  $m : C(G \times G) \to C(G)$, $m(f)(g) = f(g, h)$ and
  $u : \mathbb{C} \to C(G)$ is the unit map.

Compact Quantum Groups: Motivation

Definition

A Hopf $*$-algbra is a pair $(\mathcal{A}, \Delta)$ where
$\mathcal{A}$ is a unital $*$-algebra and we have the homomorphisms

- $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfying $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.
- $\epsilon : \mathcal{A} \to \mathbb{C}$ satisfies $(\epsilon \otimes id) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$.
- $S : \mathcal{A} \to \mathcal{A}$ satisfies $m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$ where $m : C(G \times G) \to C(G)$, $m(f)(g) = f(g, h)$ and $u : \mathbb{C} \to C(G)$ is the unit map.

Compact Quantum Groups: Motivation

G: compact group
$\mathcal{A} = C(G)$ with unital $*$-homomorphisms

$$\Delta : C(G) \to C(G \times G)$$
$$\big(\Delta(f)\big)(g, h) = f(g \cdot h)$$

Here, the comultiplication operator Δ captures the operation of the group. What does this look like in the noncommutative case?

Compact Quantum Groups: Definition

Definition (Woronowicz)

A compact quantum group (c.q.g.) is a unital C^*-algebra $\mathcal{A}$ equipped with a unital $*$-homomorphsim called comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that

- $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ as homomorphisms $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ (called coassociativity)
- the spaces $\mathrm{span}\{(a \otimes 1)\Delta(b) \mid a, b \in \mathcal{A}\}$ and $\mathrm{span}\{(1 \otimes a)\Delta(b) \mid a, b \in \mathcal{A}\}$ are dense in $\mathcal{A} \otimes \mathcal{A}$ (called the cancellation property)

Compact Quantum Groups

Coassociativity: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

Cancellation: $(\mathcal{A} \otimes 1)\Delta(\mathcal{A})$ and $(1 \otimes \mathcal{A})\Delta(\mathcal{A})$ are dense in $\mathcal{A} \otimes \mathcal{A}$

Theorem / Example

G: compact group

Then $\mathcal{A} = C(G)$ is a c.q.g. with comultiplication given by

$$\Delta : C(G) \to C(G \times G) \cong C(G) \otimes C(G)$$
$$(\Delta(f))(g, h) = f(g \cdot h)$$

Theorem / Example

Conversely, every c.q.g. $(\mathcal{A}, \Delta)$ with $\mathcal{A}$ commutative is of the form $\mathcal{A} = C(G)$ for some compact group G.

Classical Symmetry

Symmetries of a structure X are viewed as transformations of X preserving its relevant properties.

metric space

If (X, d) is a metric space then we require that the transformations do not change the metric, and we get isometries. The isometry group (symmetry group of a metric space) is the group of all bijective isometries from X onto itself, with the group operation being function composition.

The Compact Quantum Group S_n^+

Definition

A magic unitary matrix over a $*$-algebra $\mathcal{A}$ is some $n \times n$ matrix $U = [u_{xy}]_{x,y=1,\ldots,n}$ with entries $u_{xy} \in \mathcal{A}$ that satisfies

- $u_{xy} = u_{xy}^* = u_{xy}^2$
- $\sum_{y=1}^n u_{xy} = 1 = \sum_{x=1}^n u_{xy}$

Note: $\mathcal{A} = \mathbb{C} \leftrightarrow U \in M_n(\mathbb{C}) \leftrightarrow$ permutation matrix

The Compact Quantum Group S_n^+

Then $C(S_n)$ is isomorphic to the universal C^* algebra

$$C^*\Big(u_{xy} \mid U := [u_{xy}] \text{ is an } n \times n \text{ magic unitary matrix}, u_{xy} \text{ commute}\Big)$$

S_n is the *symmetry group* of a finite set with no extra structure.

Definition

We define the compact quantum group S_n^+ by the universal C^*-algebra

$$C(S_n^+) := C^*\Big(u_{xy} \mid U := [u_{xy}] \text{ is an } n \times n \text{ magic unitary matrix}\Big)$$

S_n^+ is the *quantum symmetry group* of a finite set with no extra structure.

The Compact Quantum Group S_n^+

Definition

We define the compact quantum group S_n^+ by the universal C^*-algebra

$$C(S_n^+) := C^*\left(u_{xy} \mid U := [u_{xy}] \text{ is an } n \times n \text{ magic unitary matrix}\right)$$

Then $C(S_n^+)$ becomes a c.q.g. with comultiplication $\Delta : C(S_n^+) \to C(S_n^+) \otimes C(S_n^+)$ defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

The Compact Quantum Group S_n^+

For $n = 1, 2, 3,\ C(S_n^+) = C(S_n)$.

For $n \geq 4,\ C(S_n^+) \neq C(S_n)$
i.e. $C(S_n^+)$ is non-commutative in these cases.

Indeed, for any pair of projections $p, q \in B(\mathcal{H})$, the following matrix is a magic unitary:

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

Quantum Metric Space

Quantum Metric Space

Let (X, d) be a finite metric space.
Family of relations given by $R_t = \{(x, y) \in X \times X \mid d(x, y) \leq t\}$.

$d(x, y) = 0 \Leftrightarrow x = y$	$\leftrightarrow$	R_0 is the diagonal relation	$\leftrightarrow$	$I \in \mathcal{V}_0$
$d(x, y) = d(y, x)$	$\leftrightarrow$	$R_t = R_t^T$	$\leftrightarrow$	$\mathcal{V}_t^* = \mathcal{V}_t$
$d(x, z) \leq d(x, y) + d(y, z)$	$\leftrightarrow$	$R_s R_t \subseteq R_{s+t}$	$\leftrightarrow$	$\mathcal{V}_s \mathcal{V}_t \subseteq \mathcal{V}_{s+t}$

Definition (Kuperberg-Weaver)

A quantum metric space of a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a one-parameter family of weak* closed operator systems $\mathcal{V}_t \subseteq \mathcal{B}(\mathcal{H})$, $t \in [0, \infty)$ s.t.

1. $\mathcal{V}_s \mathcal{V}_t \subseteq \mathcal{V}_{s+t}$ for all $s, t \geq 0$
2. $\mathcal{V}_t = \cap_{s>t} \mathcal{V}_s$ for all $t \geq 0$
3. $\mathcal{V}_0 = \mathcal{M}'$ where $\mathcal{M}'$ is the commutant of $\mathcal{M}$

Quantum Metric Space

Quantum metric: 1.$\mathcal{V}_s^X \mathcal{V}_t^X \subseteq \mathcal{V}_{s+t}^X$, 2.$\mathcal{V}_t^X = \cap_{s>t} \mathcal{V}_s^X$, 3.$\mathcal{V}_0^X = \mathcal{M}'$

classical case

Let (X, d) be a (classical) finite metric space.
Take the algebra $\mathcal{M} = \ell^\infty(X)$ of bounded multiplication operators on $\mathcal{H} = \ell^2(X)$, where $\mathcal{H}$ has standard basis $\{e_x\}_{x \in X}$.
i.e., $f \in \mathcal{M} \subseteq B(\ell^2(X))$ acts as $(M_f g)(x) = f(x)g(x)$.
We define $\mathcal{V}_t^X$ by

$$\mathcal{V}_t^X := \{A \in B(\ell^2(X)) \mid \langle Ae_y, e_x\rangle = 0 \text{ if } d(x,y) > t\}$$

$$= \mathrm{span}\{V_{xy} \in \mathcal{B}(\ell^2(X)) \mid d(x,y) \leq t\} \text{ where } V_{xy} : g \mapsto \langle g, e_y\rangle e_x$$

This gives us a quantum metric space in the sense of K-W

Quantum Metric Space

Quantum metric: 1. $\mathcal{V}_s^X\mathcal{V}_t^X \subseteq \mathcal{V}_{s+t}^X$, 2. $\mathcal{V}_t^X = \cap_{s>t}\mathcal{V}_s^X$, 3. $\mathcal{V}_0^X = \mathcal{M}'$

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$$= \operatorname{span}\{V_{xy} \in \mathcal{B}(\ell^2(X)) \mid d(x,y) \le t\} \text{ where } V_{xy} : g \mapsto \langle g, e_y\rangle e_x$$

This gives us a quantum metric space in the sense of K-W.

0. It's easy to see $\mathcal{M}$ is a von Neumann algebra.
1. For any V_{xy}, V_{zw}, $V_{xy}V_{zw} = \delta_{yz}V_{xw} \in \mathcal{V}_{s+t}^X$.
2. If $V_{xy} \in \mathcal{V}_s^X$ for all $s > t$, then $d(x,y) \le t$ so $V_{xy} \in \mathcal{V}_t$.
3. $\mathcal{V}_0^X = \operatorname{span}\{V_{xx}\} = \ell^\infty(X)$. Conversely, if $M \in \mathcal{M}'$ ...

Quantum Metric Space

Theorem

Conversely, if $\mathcal{V}_t$ is a quantum metric on $\mathcal{M} = \ell^\infty(X)$, then

$$d(x, y) := \inf\{t \mid \exists A \in \mathcal{V}_t \text{ s.t. } \langle Ae_y, e_x\rangle \neq 0\}$$

is a metric on X.

Banica's quantum isometry group

Definition (Banica)

Take a (classical) finite metric space (X, d) with $|X| = n$ and let $D = [d(x,y)]_{x,y\in X}$ be the $n \times n$ distance matrix.
The quantum isometry group (or quantum symmetry group of the metric space) to be

$$G^+(X,d) = C(S_n^+) \ / \ \langle UD = DU \rangle$$

i.e. the quotient of $C(S_n^+)$ by the ideal generated by the relations $UD = DU$.

Goal: capture Banica's definition G^+ using K-W quantum metric spaces

Definition

Consider two q. metric space $(\mathcal{M}_1, \mathcal{V}_t)$ and $(\mathcal{M}_2, \mathcal{W}_t)$ with fixed ONB bases $\{e_j\}$ for $\mathcal{M}_1$ and $\{f_k\}$ for $\mathcal{M}_2$.
We define the quantum isometry group between quantum metric spaces, $G^{\mathcal{V},\mathcal{W}}$, to be the universal c.q.g. generated by $P = [p_{ij}] \in C(G^{\mathcal{V},\mathcal{W}}) \otimes B(\mathcal{H}_1, \mathcal{H}_2)$ giving a unital $*$-homomorphism

$$\delta_{\mathcal{V},\mathcal{W}} : \mathcal{M}_1 \to \mathcal{M}_2 \otimes C(G^{\mathcal{V},\mathcal{W}})$$
$$e_j \mapsto \sum_k f_k \otimes p_{kj}$$

and ensuring the conjugation map given by

$$\alpha_{\mathcal{V},\mathcal{W}} : B(\mathcal{H}_1) \to B(\mathcal{H}_2) \otimes C(G^{\mathcal{V}})$$
$$T \mapsto P(T \otimes 1)P^*$$

leaves $\mathcal{V}_t$ invariant, i.e. $\alpha_{\mathcal{V},\mathcal{W}}(\mathcal{V}_t) \subseteq \mathcal{W}_t \otimes C(G^{\mathcal{V},\mathcal{W}})$ for all t.

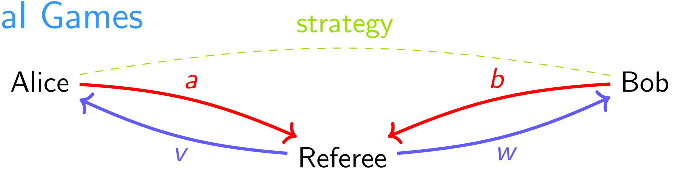
Theorem (E. '20)

Let (X, d) be a (classical) finite metric space, set $\mathcal{M} = \ell^\infty(X)$, and let $\mathcal{H} = \ell^2(X)$. Let $\mathcal{V}_t^X$ be the standard construction of the K-W quantum metric.

Then $G^+(X, d) \cong G^{\mathcal{V},\mathcal{V}}$.

Non-local Games

Non-local Games

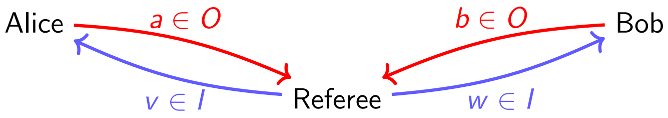


The players, Alice and Bob, know the game they are playing and may agree on a strategy before gameplay begins.
One round of the game will look like:

- the referee asks questions v and w to Alice and Bob, respectively
- without communicating, Alice and Bob reply with a and b
- the referee determines if Alice and Bob win that round

Goal: Win every round of the game

Non-local Games



The synchronous non-local game is given by $G = (I, O, \lambda)$.
The set I represents the inputs (questions) that the players Alice and Bob can receive. The set O represents the outputs (answers) that Alice and Bob can produce.
The rules of the game are represented by the function

$$\lambda : I \times I \times O \times O \to \{0, 1\}$$

They win the game if $\lambda(v, w, a, b) = 1$ and lose otherwise.
Each game must satisfy $\lambda(v, v, a, b) = \delta_{a,b}$.

Game Strategies

A deterministic strategy for a game is a function $h : I \to O$ such that if Alice (or Bob) receives input v, they answer output $h(v)$.

A strategy is called random if during different rounds of the game, Alice and Bob receive inputs v and w and answer different outputs. We may observe the game and get the conditional probabilitiess $p(a, b|v, w)$.

A strategy is perfect if $\lambda(v, w, a, b) = 0 \Rightarrow p(a, b|v, w) = 0$.

The Metric Isometry Game

Let (X, d_X) and (Y, d_Y) be two metric spaces, $X \sqcup Y = I = O$. The rule function satisfies $\lambda(x_A, x_B, y_A, y_B) = 1$ if and only if the following conditions are met:

- x_A and y_A belong to different spaces
- x_B and y_B belong to different spaces
- if x_A and x_B are from the same space then $d_.(x_A, x_B) = d_.(y_A, y_B)$
- if x_A and x_B are from different spaces then $x_A = y_B$ if and only if $x_B = y_A$

Proposition (E.)

There exists a perfect (classical) strategy $\Leftrightarrow$ X is isometric to Y

Classical Strategy vs Quantum Strategy

A classical strategy for a non-local game is one in which the only resource available to the players is shared randomness (i.e. shared probability space).

In a quantum strategy, the players are allowed to perform local quantum measurements on a shared entangled state.

There are several different mathematical models to describe $p(a, b|x, y)$ For $|I| = n$, $|O| = k$ let $C_t(n, k)$ be the set of conditional probabilities in model t.

$$C_{classical}(n, k) \subseteq C_{qc}(n, k) \subseteq M_{nk}(\mathbb{C}).$$

qc = quantum commuting

Quantum Strategies

$$C_{classical}(n,k) \subseteq C_{qc}(n,k) \subseteq M_{nk}(\mathbb{C}).$$

- $C_{classical}(n,k)$ is the set of conditional probabilities $p(a,b|v,w)$ which arise from Alice and Bob sharing a probability space $(\Omega, \mathbf{P})$ each having random variables $f_{\omega,A}, g_{\omega,B} : I \to O$ with
$$p(a,b|v,w) = \mathbf{P}\left(\omega \in \Omega \mid f_{\omega,A}(v) = a,\ g_{\omega,B}(w) = b\right).$$
- in $C_{qc}(n,k)$, there is a Hilbert space H on which Alice and Bob are allowed to make measurements, and a shared state $\psi \in \mathcal{H}$. Alice has orthogonal projections $e_{v,a} \in B(H)$ satisfying $\sum_a e_{v,a} = \mathrm{id}_H$ (Bob has $f_{w,b} \in B(H)$ satisfying $\sum_b f_{w,b} = \mathrm{id}_H$) such that
$$p(a,b|x,y) = \langle e_{v,a} f_{w,b} \psi, \psi \rangle$$

The Game $*$-algebra

We define the $*$-algebra of a synchronous game G, $\mathcal{A}(G)$, to be defined as the quotient of the free $*$-algebra generated by $\{e_{v,a} \mid v \in I, a \in O\}$ subject to the relations

- $e_{v,a} = e_{v,a}^*$
- $e_{v,a} = e_{v,a}^2$
- $1 = \sum_a e_{v,a}$
- $e_{v,a}e_{w,b} = 0$ for all v, w, a, b such that $\lambda(v, w, a, b) = 0$

The Game $*$-algebra

Theorem (Helton-Meyer-Paulsen-Satriano, Kim-Paulsen-Schafhauser)

For a synchronous game G,

- G has a perfect deterministic strategy $\Leftrightarrow$ G has a perfect classical strategy $\Leftrightarrow$ there exists a unital $*$-homomorphism from $\mathcal{A}(G)$ to $\mathbb{C}$
- G has a perfect qc-strategy $\Leftrightarrow$ there exists a unital C^*-algebra $\mathcal{C}$ with a faithful trace and a unital $*$-homomorphism $\pi : \mathcal{A}(G) \to \mathcal{C}$

The Game $*$-algebra

ex.

Consider the metric isometry game.
Then the game $*$-algebra for the metric isometry game, $\mathcal{A}(Isom(X, Y))$, is the $*$-algebra generated by $\{e_{x,y} \mid x \in X, y \in Y\}$ subject to the relations that $U = [e_{x,y}]$ is a magic unitary matrix with

$$(1 \otimes D_X)U = U(1 \otimes D_Y)$$

Theorem (E.)

For classical metric spaces (X, d_X) and (Y, d_Y), and their corresponding quantum metric spaces $(\ell^\infty(X), \mathcal{V}_t)$ and $(\ell^\infty(Y), \mathcal{W}_t)$, we have $C(G^{\mathcal{V},\mathcal{W}}) = \mathcal{A}(Isom(X, Y))$.

The Game $*$-algebra

Theorem (E.)

Given two classical metric spaces (X, d_X) and (Y, d_Y), TFAE:

- $\mathcal{A}(Isom(X, Y)) \neq 0$
- $\mathcal{A}(Isom(X, Y))$ admits a non-zero C^*-representation
- The metric isometry game has a perfect quantum-commuting (qc)-strategy, $X \cong_{qc} Y$

Thank you!