Quantum Graphs!

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Quantum Symmetries Student Seminar

Feb 12, 2021

Outline of my talk

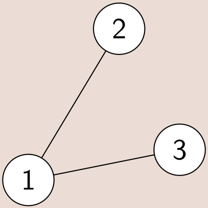
1. The quantum analogue of a graph
2. Significance of quantum graphs in information theory
3. Different approaches to quantum graphs
4. Quantum coloring problem

Classical Graphs

G = (Vertex set, Edge set, Adjacency matrix)

Example of a classical graph

- $V = \{1, 2, 3\}$
- $E = \{(1,2), (1,3)\}$
- $A_G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$

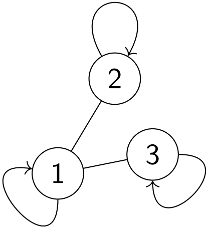


$$S_G := \left\{ \begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \text{ where } * \in \mathbb{C} \right\} \subseteq M_3(\mathbb{C})$$

S_G is a subspace !

When the graph is reflexive....

$$S_G := \left\{ \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \text{ where } * \in \mathbb{C} \right\} \subseteq M_3(\mathbb{C})$$



Properties of S_G:

- Linear subspace
- Self-adjointness ($A \in S_G \iff A^* \in S_G$)
- Contains identity

S_G is an operator system!

Operator System

A subspace $S \subseteq B(H)$ is called an operator system if

- $I \in S$.
- $A \in S \implies A^* \in S$.

Suppose G is a classical graph with vertex set $V = \{1, 2 \ldots n\}$.

Non-commutative graph associated with a classical graph

The non-commutative graph associated with the classical graph $G = (V, E)$ is the operator system S_G defined as

$$S_G = \text{ span}\{e_{ij} \ : \ (i,j) \in E \text{ or } i = j, \ \ \forall i,j \in V\} \subseteq \mathbb{M}_n \ ,$$

where e_{ij} are matrix units in $\mathbb{M}_n$.

More generally,

Matrix Quantum Graphs

An operator system in $\mathbb{M}_n$ is called a Matrix quantum graph.

Motivation from Information theory

- Matrix quantum graphs generalize the confusability graph of classical channels.
- Confusability graphs $->$ zero-error classical communication.
- Quantum graphs $->$ analogous role in zero-error quantum communication.

Classical Channel

$\Phi \longleftrightarrow$ Probability transition function $[P(y|x)]$.

$$\text{(Input messages) } X \xrightarrow{\Phi} Y \text{ (Output messages)}$$
$$\{x_1, x_2 \ldots x_m\} \xrightarrow{\Phi} \{y_1, y_2 \ldots y_n\}$$

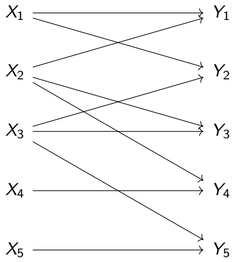
Confusability graph of classical channel

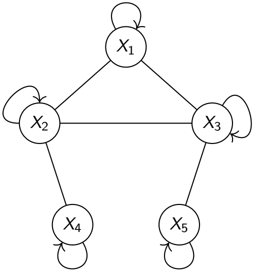
$(\Phi : X \to Y) \longleftrightarrow$ Probability transition function $[P(y|x)]$.

Confusability graph of Φ

- Vertex set: $X = \{x_1, x_2 \ldots x_m\}$.
- Edges: $x_i \sim x_j$ if there exists $y \in Y$ such that $P(y|x_i)P(y|x_j) > 0$.

Input messages $(X) \xrightarrow{\Phi}$ Output messages(Y)





Quantum Channels

- Quantum communication channel take quantum states to quantum states.

$$\Phi : B(H_A) \xrightarrow{\text{linear}} B(H_B)$$

TP : Trace preserving: $\mathrm{Tr}(\rho) = \mathrm{Tr}(\Phi(\rho))$.

CP : Completely positive: Φ is positive and all extensions $\Phi \otimes I_E$ are also positive.

- CPTP maps have several representations :

Kraus form

$\Phi(\rho) = \sum_{i=1}^{r} K_i \rho K_i^*$, where $K_i \in B(H_A, H_B)$ satisfying $\sum_{i=1}^{r} K_i^* K_i = I_A$.

- The Kraus operators are not unique.

Classical embedded in Quantum

Input $A = \{1, 2 \dots m\} \longrightarrow B = \{1, 2 \dots n\}$ Output

CLASSICAL	QUANTUM
Input: $\lvert i \rangle = e_i \in \mathbb{C}^m$	Input: matrix units $e_{ii} \in \mathbb{M}_m$ ($e_{ii} = \lvert i \rangle \langle i \rvert$)
Output: $\lvert j \rangle = e_j \in \mathbb{C}^n$	Output: matrix units $e_{jj} \in \mathbb{M}_n$ ($e_{jj} = \lvert j \rangle \langle j \rvert$)
$\mathbb{C}^m \xrightarrow{\Phi} \mathbb{C}^n$ $\Phi(v) = Pv$, where $P = [P(b\lvert a)]_{a \in A, b \in B}$	$\mathbb{M}_m \xrightarrow{\Phi} \mathbb{M}_n$ $\Phi(X) = \sum_{a \in A, b \in B} K_{ab}(X) K_{ab}^*$, where Kraus operators $K_{ab} = \sqrt{P(b\lvert a)}\, e_{ba} \in \mathbb{M}_{n \times m}$
Confusability graph G $a \sim c \iff \exists\, b$ with $P(b\lvert a) P(b\lvert c) \neq 0$	$K_{ab}^* K_{cd} = \sqrt{P(b\lvert a) P(d\lvert c)}\, \delta_{bd} e_{ac}$ $K_{ab}^* K_{cd} \neq 0 \iff$ $b = d$ and $P(b\lvert a) P(d\lvert c) \neq 0$
$S_\Phi = \mathrm{span}\{e_{ac} : a \sim c\}$	$S_\Phi = \mathrm{span}\{K_{ab}^* K_{cd} : a, c \in A$ and $b, d \in B\}$

Quantum Graphs

Non-commutative confusability graph [DSW, 2013]

Given a quantum channel $\Phi : \mathbb{M}_m \to \mathbb{M}_n$ with $\Phi(x) = \sum_{i=1}^{r} K_i x K_i^*$, the confusability graph of Φ is the operator system:

$$S_\Phi = \text{ span}\{K_i^* K_j : 1 \leq i, j \leq r\} \subseteq \mathbb{M}_m.$$

This is independent of the Choi-Kraus representation of Φ.

Every operator system arises from a quantum channel!

Proposition

Let $S \subseteq \mathbb{M}_m$ be an operator system. Then there is $n \in \mathbb{N}$ and a quantum channel $\Psi : \mathbb{M}_m \to \mathbb{M}_n$ such that $S = S_\Psi$.

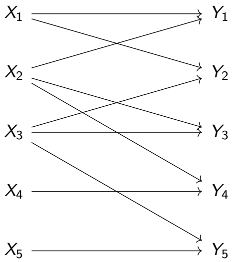
Applications in zero-error communication

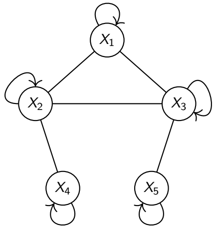
Goal

Send messages through a channel without confusion.

- Classical: x_i and x_j are not confusable $\iff$ $x_i \not\sim x_j$ in the confusability graph.

Input messages $(X) \xrightarrow{\Phi}$ Output messages(Y)





- One-shot zero error capacity of ϕ = Independence number of G = maximum number of messages transmitted without confusion.

Zero-error quantum communication

- Quantum states: $\rho, \sigma \in B(H)$ are distinguishable $\iff \langle \rho, \sigma \rangle = 0$.

$$\Phi(\rho) = \sum_{i=1}^{r} K_i \rho K_i^*, \quad S_\Phi := \mathrm{span}\{K_j^* K_i : 1 \leq i, j \leq r\}.$$

- Encode input message $x \mapsto \rho_x = |x\rangle \langle x| \in B(H)$.
- ρ_x, ρ_y are not confusable $\iff \Phi(\rho_x), \Phi(\rho_y)$ are distinguishable.
- $\langle \Phi(\rho_x), \Phi(\rho_y) \rangle = 0$, with respect to Hilbert-Schmidt inner product.

$$\mathrm{Tr}(\Phi(\rho_y)^* \Phi(\rho_x)) = 0 \iff \sum_{i,j=1}^{r} |\langle y, K_i^* K_j x \rangle|^2 = 0$$

$$\iff \mathrm{Tr}(|x\rangle \langle y| K_i^* K_j) = 0 \iff (|x\rangle \langle y|) \perp K_j^* K_i, \quad \forall i, j.$$

Result

Input messages x, y are not confusable $\iff |x\rangle \langle y| \perp S_\Phi$.

Other approaches to quantum graphs

Classical graph $G = (V, E, A_G)$

- Quantize confusability graph of classical channels [DSW, 2010]
  - Matrix quantum graphs and Operator systems
  - Projection P_S onto the operator system S
- Quantize edge set $E \subseteq V \times V$ [Weaver, 2010, 2015]
  - Quantum relations
  - Projection P_E from χ_E
- Quantize adjacency matrix [MRV, 2018]
  - Categorical theory of quantum sets and quantum functions
  - Projection P_G using A_G

Unification

Under appropriate identifications, range of these projections is the same operator system!

Quantum Relations

Quantum set: von-Neumann algebra $\mathcal{M} \subseteq B(H)$

$$\mathcal{M}' := \{A \in B(H) \text{ such that } AM = MA, \ \forall \, M \in \mathcal{M}\}.$$

Quantum relation [Weaver, 2010]

A quantum relation on $\mathcal{M}$ is a weak*-closed subspace $S \subseteq B(H)$ that is a bi-module over its commutant $\mathcal{M}'$, i.e. $\mathcal{M}' S \mathcal{M}' \subseteq S$.

- Independent of the representation $\mathcal{M} \subseteq B(H)$.
- Quantum relations on $l^\infty(V) \longleftrightarrow$ subsets of $V \times V \longleftrightarrow$ relations on V.
- S contains operators that "connect adjacent vertices".

Quantum graphs as quantum relations

Classical graph: $E \subseteq V \times V$ - reflexive, symmetric relation on V.

Quantum Graph [Weaver, 2015]

A quantum graph on $\mathcal{M}$ is a reflexive and symmetric quantum relation on $\mathcal{M}$.

Quantum relation $S \subseteq B(H)$ on $\mathcal{M}$ is:

- Reflexive $\iff \mathcal{M}' \subseteq S$ ($\implies 1 \in S$).
- Symmetric $\iff S^* = S$.

Connection to operator system

Quantum graph S is a weak*-closed operator system that is a bimodule over $\mathcal{M}'$.

Projection picture

Motivation from commutative setting:

Classical graph $G = (V, E)$ with vertex set V and edge set $E \subseteq V \times V$:

$$\begin{array}{ccc} \chi_E \in C(V \times V) & \cong & C(V) \otimes C(V) \\ \chi_E & \longleftrightarrow & \sum_{x,y \in V} \delta_{xy} \, (\chi_x \otimes \chi_y) \end{array}$$

where $\delta_{xy} = 1$ if $(x, y) \in E$ and 0 otherwise.

Properties

- Idempotent: $\chi_E = \chi_E^* = \chi_E^2$
- Reflexive: $m(\chi_E) = 1_V$
- Symmetric: $\sigma(\chi_E) = \chi_E$

where $m : C(V) \otimes C(V) \xrightarrow{\text{multiply}} C(V)$ and $\sigma : C(V) \otimes C(V) \xrightarrow{\text{swap}} C(V) \otimes C(V)$.

Quantum graph as projections

Quantum set: finite dimensional C*-algebra M with fixed tracial state.

Definition

A quantum graph is a quantum set $M \subseteq B(H)$ with a projection $p \in M \otimes M^{op}$ satisfying

- $p = p^* = p^2$
- $m(p) = 1_M$
- $\sigma(p) = p$

$$p \in M \otimes M^{op} \underset{\pi}{\cong} {}_{M'}CB_{M'}\big(B(H)\big)$$

Connection to operator system

$S := \mathrm{Range}(\pi(p)) \subseteq B(H)$ is a weak*-closed operator system in $B(H)$ that is a bimodule over M'.

Quantizing adjacency matrix...

Definition

A quantum graph is a pair (M, A_G) *containing*

- *Quantum set* M
- *Quantum adjacency matrix* $A_G : M \xrightarrow{linear} M$ *with*
  - *Idempotency:* $m(A_G \otimes A_G)m^* = A_G$
  - *Reflexivity:* $m(A_G \otimes I)m^* = I$
  - *Symmetry:* $(\eta^* m \otimes I)(I \otimes A_G \otimes I)(I \otimes m^* \eta) = A_G$

Back to projections: Get $p \in M \otimes M^{op}$ as

$$p := (I \otimes A_G)m^* \eta.$$

Advantage of quantum adjacency matrix

Allows us to define the spectrum of a quantum graph!

Comparing different notions of quantum graphs

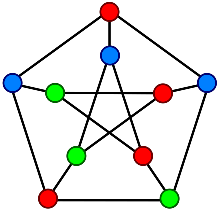
Quantum set M: finite dimensional C*-algebra with fixed tracial state ψ.

CLASSICAL GRAPH	MATRIX Q.GRAPH	QUANTUM RELATIONS	PROJECTIONS	ADJACENCY MATRIX
$G = (V, E, A_G)$ $A_G \in \mathbb{M}_n\{0,1\}$	$S \subseteq \mathbb{M}_n$ is an operator system.	$(M, {}_{M'}S_{M'})$ weak*-closed operator sys in $B(H)$, bimodule over M'.	(M, p) $p \in M \otimes M^{op}$	(M, A_G) $A_G : M \to M$
Idempotency: $A_G \odot A_G = A_G$	$A_G \odot (\mathbb{M}_n) = S$	$M'SM' \subseteq S$	$p = p^* = p^2$	$m(A_G \otimes A_G)m^* = A_G$
Reflexivity: 1s on the diagonal	$1 \in S$	$M' \subseteq S$	$m(p) = 1_M$	$m(A_G \otimes I)m^* = I$
Undirected: $A_G = A_G^T$	$S = S^*$	$S = S^*$	$\sigma(p) = p$	$(\eta^* m \otimes I)(I \otimes A_G \otimes I)(I \otimes m^* \eta) = A_G$

Graph coloring

My research

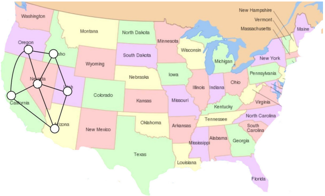
Assign colors to vertices of graph such that no adjacent vertices get same color.



Chromatic number

Least number of colors required to color that graph.

Quantum Coloring Problem

<table>
<tr><td></td><td>Classical
Graph</td><td>Quantum
Graph</td></tr>
<tr><td>Classical
Chromatic No.</td><td>
</td><td>Partition orthonormal basis into independent subsets</td></tr>
<tr><td>Quantum
Chromatic No.</td><td>

Non-Local Graph Coloring Game</td><td></td></tr>
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Non-local graph coloring game

We begin with a classical graph $G = (V, E)$.

The referee sends questions (vertices) to Alice and Bob separately. They respond with answers (colors), without communicating with one another.

- Inputs: $I_{alice} = I_{bob} = V$.
- Outputs: $O_{alice} = O_{bob} = \{1, 2, 3 \dots k\}$
- Rule function $\lambda : I_{alice} \times I_{bob} \times O_{alice} \times O_{bob} \longrightarrow \{0, 1\}$.
- Winning condition: $\lambda(v, w, a, b) = 1$
  - Adjacency rule: $(v, w) \in E \implies a \neq b$
  - Same vertex rule: $v = w \implies a = b$

Classical Coloring of a Matrix Quantum Graph

Definition

Let $S \subseteq \mathbb{M}_n$ be an operator system. We say there is a k-coloring of S if there is an orthonormal basis $\{v_1, v_2 \ldots v_n\}$ for $\mathbb{C}^n$ and a partition of $\{1, 2 \ldots n\}$ into k subsets $S_1, S_2 \ldots S_k$ such that

$$|v_i\rangle \langle v_j| \perp S, \text{ for all } v_i, v_j \in S_l, \text{ with } i \neq j.$$

Quantum Edge Basis

1. Let $(S, \mathcal{M}, M_n)$ be a quantum graph.
2. Let $K_1, K_2, \ldots K_m$ be non-zero subspaces of $\mathbb{C}^n$ with

$$K_1 \oplus K_2 \oplus \ldots K_r = \mathbb{C}^n,$$

   such that $\mathcal{M}$ acts irreducibly on each K_r.
3. Let E_r be the orthogonal projection of $\mathbb{C}^n$ onto K_r, $\quad 1 \leq r \leq m$.

There exists an orthonormal basis $\mathcal{F}$ of S with respect to the unnormalized trace, such that

- $\frac{1}{\sqrt{\dim(K_r)}} E_r \in \mathcal{F}$ for each $1 \leq r \leq m$;
- $\mathcal{F}$ contains an orthonormal basis for M'; and
- For each $Y \in \mathcal{F}$, there are unique r, s with $E_r Y E_s = Y$.

The quantum-to-classical graph coloring game

1. Let $(S, \mathcal{M}, M_n)$ be a quantum graph.
2. $K_1 \oplus K_2 \oplus \ldots K_r = \mathbb{C}^n$ and $\mathcal{M}$ acts irreducibly on each K_j.
3. $\{v_1, v_2, \ldots v_n\} \underset{basis}{\subseteq} \mathbb{C}^n$ that can be partitioned into bases for $\{K_i\}_i^r$.

Definition (BGH, 2020)

The **quantum-to-classical graph coloring game** *for* $(S, \mathcal{M}, M_n)$*, with respect to the basis* $\{v_1, ..., v_n\}$ *and a quantum edge basis* $\mathcal{F}$ *for* S *is:*

- *Inputs:* $\sum_{p,q} y_{\alpha,pq} \; v_p \otimes v_q$*, where* $Y_\alpha := \sum_{p,q} y_{\alpha,pq} \; v_p v_q^* \in \mathcal{F}$*.*
- *Outputs: colors* $\{1, 2, \ldots k\}$*.*
- *Winning Criteria:*
  - *Adjacency rule: If* $Y_\alpha \perp \mathcal{M}'$*, then respond with different colors.*
  - *Same vertex rule: If* $Y_\alpha \in \mathcal{M}'$*, then respond with the same color.*

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THANK YOU FOR YOUR ATTENTION!