Exercises

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Entropy

Definition. Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. If μ is another measure on $(\Omega, \mathcal{F})$, we define

$$h(\mu \| \nu) := -\int \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} \, d\nu$$

if μ is absolutely continuous with respect to ν, and $h(\mu\|\nu) = -\infty$ otherwise.

Exercise 1. If μ and ν are probability measures, then $h(\mu\|\nu) \leq 0$. *Hint: Apply Jensen's inequality to $-t \log t$.*

Exercise 2. Let $V : \mathbb{R}^d \to \mathbb{R}$ such that V is bounded below and $\int e^{-V} \, dx = 1$. Let dx denote Lebesgue measure on $\mathbb{R}^d$ with their respective Borel σ-algebras. If μ is a Borel probability measure, then

$$h(\mu \| e^{-V} \, dx) = h(\mu \| dx) - \int V \, d\mu,$$

provided that at least one of the terms on the right-hand side is finite.

Exercise 3. For a Borel probability measure μ on $\mathbb{R}^d$,

$$h(\mu \| dx) \leq \frac{1}{2} \int x^2 \, d\mu(x).$$

Definition. Let Ω and Ω' be metric spaces equipped with the Borel σ-algebra. If μ is a probability measure on Ω and $f : \Omega \to \Omega'$ is continuous, we define $f_* \mu$ by

$$f_* \mu(E) = \mu(f^{-1}(E)).$$

Exercise 4. Let f be a diffeomorphism of $\mathbb{R}^d$ and let $V : \mathbb{R}^d \to \mathbb{R}$ be a function such that $\int e^{-V} \, dx = 1$. Then

$$f_*(e^{-V} \, dx) = e^{-(V \circ f^{-1} - \log |\det Df^{-1}|)} \, dx.$$

Exercise 5. Let f be a diffeomorphism of $\mathbb{R}^d$ and μ a Borel probability measure on $\mathbb{R}^d$. Then

$$h(f_* \mu \| dx) = h(\mu \| dx) + \int \log |\det Df| \, d\mu.$$

Upper semicontinuity

Definition. Let Ω be a topological space. A function $f : \Omega \to [-\infty, \infty]$ is said to be *upper semi-continuous* if $f^{-1}([-\infty, t))$ is open for every $t \in \mathbb{R}$.

Exercise 6. A function $f : \Omega \to [-\infty, \infty]$ is upper semi-continuous if and only if, for every net $(x_i)_{i\in I}$ converging to a point x, we have

$$\limsup_{i\in I} f(x_i) \leq f(x).$$

Exercise 7. If $\mathcal{F}$ is a family of upper semi-continuous functions $\Omega \to [-\infty, \infty]$, then

$$g(x) := \inf_{f\in\mathcal{F}} f(x)$$

is upper semi-continuous.

Exercise 8. Let $a, b \in [-\infty, \infty]$ with $a < b$. Let $E \subseteq \Omega$, and let

$$f(x) = \begin{cases} a, & x \in E, \\ b, & x \in \Omega \setminus E. \end{cases}$$

Then f is upper semi-continuous if and only if E is open.

Exercise 9. If $f : \Omega \to [-\infty, \infty]$ is upper semi-continuous and $K \subseteq \Omega$ is compact, then f achieves a maximum on K.