Symmetric Trivalent Categories

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February 25, 2021

Planar diagrams

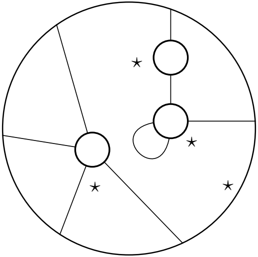
Definition

A planar diagram consists of the following data:

1. An outer disc and a finite number of inner discs.
2. A finite number of marked boundary points on both outer and inner discs.
3. A collection of smooth, non-crossing curves called strings whose endpoints are the boundaries of the discs.
   - If there are $2n$ total (inner and outer) boundary points, the planar diagram will have n strings connecting those boundary points.
   - The strings lie between the internal discs and external disc.
4. The outer disc and each inner disc will have a starred region.

Example diagram

Diagrams are considered up to smooth isotopy. A planar diagram looks like

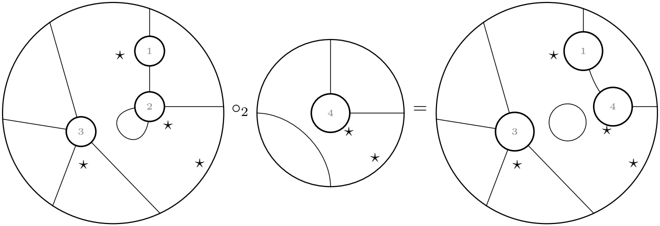


(1)

Note that these diagrams are sometimes called planar tangles, but this notation will be avoided due to the planar algebra we are working with.

Composition of tangles

We can compose diagram A and B if the number of strands connecting to the outer boundary of B is equal to the number of strands connecting to one of the inner boundaries of A. An example of this composition is below:



(2)

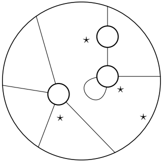
We call the set of isotopy classes of planar diagrams with the compositions described above the planar operad, $\mathcal{P}$.

Planar Algebra Definition

Definition

A planar algebra is a family of vector spaces $\{V_k\}_{k=0,1,2,3,\ldots}$ together with an action of the planar operad.

In (1), for example, a planar algebra would determine a multilinear map



$: V_2 \otimes V_4 \otimes V_4 \to V_6$

that respects any composition like (2).

Planar algebra definition

Some planar algebras will also have the following nice property:

Definition

A planar algebra is said to have modulus d if inserting a closed string inside a planar diagram causes its multilinear map to be multiplied by d.

Definition

A homomorphism of planar algebras is a collection of linear maps $\{\phi_k : V_k \to W_k\}$ that intertwine the two actions of $\mathcal{P}$. We say that two planar algebras V and W are isomorphic if the $\{\phi_k\}$ are isomorphisms.

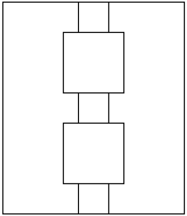
Standard form

Often we draw planar diagrams in “standard form” as follows:

1. Draw all boundaries (inner and outer) as rectangles parallel to the axes.
2. Orient rectangle such that the sides have no strings and the left side of the rectangle is the starred region.
3. Typically, half of strings will go into the top of the diagram and half will go into the bottom.
4. All strings are drawn smoothly.

Standard form example

- As an example consider the following diagram, which is commonly called the “multiplication” diagram



- Essentially, multiplying two diagrams in a planar algebra is the same as vertically stacking them.

Skein Theories

- A natural class of examples of planar algebra comes from tangles as well as graphs.
- When one defines a ring or algebra, we often describe it using generators and relations.

Definition

A planar algebra, $\mathcal{V}$, is generated by a set of elements, $\mathcal{R}$, if for every element $v \in \mathcal{V}$, there exists a planar diagram, T, such that $T(r_1, \ldots, r_k) = v$ for some $r_i \in \mathcal{R}$.

Skein Theories

- When one studies all quotients of a ring or algebra, one might instead classify all two-sided ideals.

Definition

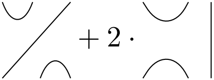
Let $\mathcal{V}$ be a planar algebra. A planar ideal, $\mathcal{I}$, is a subset of $\mathcal{V}$ such that $\mathcal{I}$ is the kernel of some quotient map of $\mathcal{V}$.

- Equivalently, $\mathcal{I}$ is a planar ideal if it is closed under arbitrary composition with any element of the planar algebra.

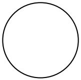
Definition

Let $\mathcal{V}$ be a planar algebra. A *skein relation* is an element of the kernel of a quotient map from $\mathcal{V}$. A *skein theory* is a set of skein relations that generates a planar ideal.

- One of the most common examples of a planar algebra is the Tempereley-Lieb-Jones planar algebra, TLJ(d).
- It is the planar algebra generated by *no* objects.
- Thus, all elements of TLJ(d) contain only non-crossing strings. For example,



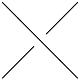
- The only relation is a circle relation:



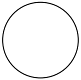
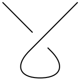
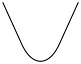
$= d$

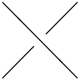
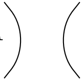
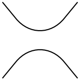
- In this way, one can think of any unshaded planar algebra with modulus d as containing TLJ(d) as a sub-planar algebra.
- The dimensions of the box spaces of TLJ(d) are indexed by the Catalan numbers $(1, 1, 2, 5, \dots)$

- Another major class of planar algebras comes from those generated by

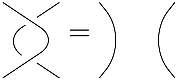


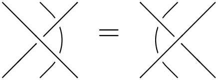
- With the following relations, TLJ(d) (with $d = -(A^2 + A^{-2})$) can be thought of as a planar algebra generated by the crossing:

 $= -(A^2 + A^{-2})$  $= -A^{-3}$ 

 $= A$  $+ A^{-1}$ 

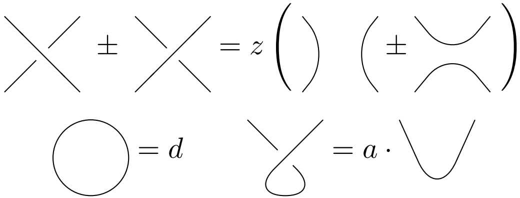
- In addition, TLJ(d) satisfies the Reidemeister moves:

Reidemeister II (R2): 

Reidemeister III (R3): 

- Note that R1 is explicitly not satisifed!
- Obviously the box-space dimensions of TLJ(d) are the same $(1, 1, 2, 5, 14 \ldots)$.

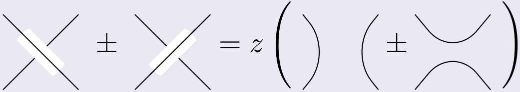
- Another common example of a planar algebra generated by a crossing is the Kauffman/Dubvronik polynomial planar algebra
- In addition to R2 and R3, it is subject to the following skein relations:



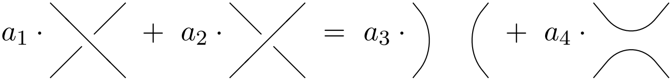
- Note that z, d, and a are not all free parameters.
- The box-space dimensions are $1, 1, 3, 15, \ldots$

Theorem

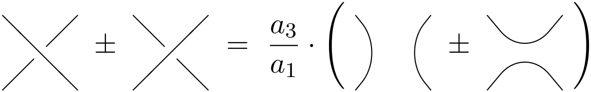
Let $\mathcal{V}$ be a quotient of the planar algebra of tangles such that $\dim V_4 = 2$ *or* 3. *Then there exists a relation of the form*



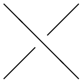
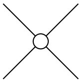
- Since $\dim V_4 = 2$ or 3, we know generically



- By rotating the equation however, we see that $a_1 = \pm a_2$ and $a_3 = \pm a_4$. We also see that $a_1 \neq 0$.
- So we can rewrite the equation as



- A consequence of this theorem is that $\mathrm{TLJ}(d)$ and the Kauffman/Dubrovnik polynomial planar algebras are the only planar algebras generated by the crossing with $\dim V_4 \leq 3$.
- It is this type of classification of "small" planar algebras that we will be concerned with today.
- One might also consider what small planar algebras exist up to isomorphism that are generated by

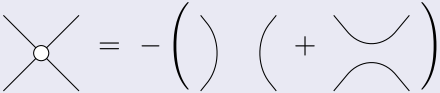
 and 

- This second crossing is called a virtual crossing.
- Can think of it as literally passing "through" the strands.

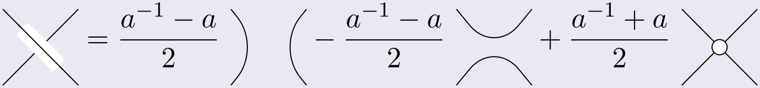
Theorem (E. '18)

The only small planar algebras generated by a crossing and a virtual crossing are isomorphic to one of the following:

1. *A virtual TLJ planar algebra*
2. *The Kauffman polynomial planar algebra at* $a = \pm 1$ *and* $d = -2$ *equipped with the virtual crossing*

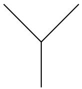


3. $Rep(O(2), a)$*, where* $a \in \mathbb{C}^\times$ *denotes that the planar algebra is equipped with a braiding given by the formula*



(3)

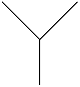
- Another class of planar algebras are those of graphs.
- In our case, we will consider graphs generated by a symmetric, trivalent vertex.



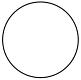
- Elements of this planar algebra can be thought of as trivalent graphs, graphs with only trivalent and univalent vertices
- Because strings cannot cross, these graphs must also be *planar*

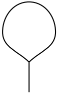
Previous work

In 2017 work of Morrison, Peters, and Snyder, all planar algebras generated by

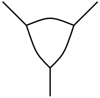
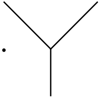


were classified with certain smallness conditions and subject to the following relations:

 $= d = t - 1$

 $= 0$

 $= c_1 \cdot$

 $= c_2 \cdot$ 

Previous work

Theorem (From MPS 2017)

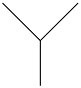
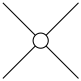
The following table is an exhaustive list of non-degenerate trivalent planar algebras with initial box space dimension bounds $1, 0, 1, 1, 4, 11, 41$*:*

Dimension bounds	*Name*
$1, 0, 1, 1, 2, \dots$	$SO(3)_{\zeta_5}$
$1, 0, 1, 1, 3, \dots$	$SO(3)_q$ *or* $OSp(1\|2)$
$1, 0, 1, 1, 4, 8, \dots$	ABA
$1, 0, 1, 1, 4, 9, \dots$	$(G_2)_{\zeta_{20}}$
$1, 0, 1, 1, 4, 10, \dots$	$(G_2)_q$
$1, 0, 1, 1, 4, 11, 37, \dots$	$H3$

where ABA *is a sub-planar algebra of the free product of* $TL(\sqrt{dt^{-1}}) * TL(t)$ *and* $H3$ *is the fusion category found in work by Grossman and Snyder 2012.*

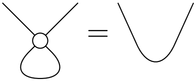
Symmetric Trivalent Categories

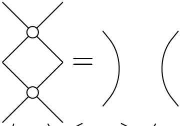
My work expands on this classification by looking at planar algebras generated by

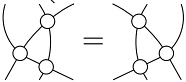
 and 

Virtual Reidemeister Moves

The virtual crossing has corresponding virtual Reidemeister moves:

Virtual Reidemeister I (vR1): 

Virtual Reidemeister II (vR2): 

Virtual Reidemeister III (vR3): 

in addition to some naturality conditions with the trivalent vertex.

Equivalently, one can think of adding the virtual crossing as simply removing the planarity conditions from the graphs.

Computer Use

Because the virtual crossing allows us to consider any graph, we can more easily utilize Mathematica to answer our questions.

```
In[]:= STtoGraph[Collect[R3L, listR32]]
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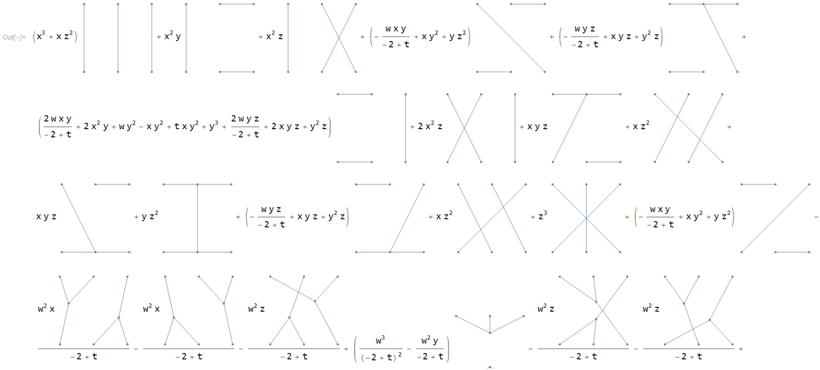


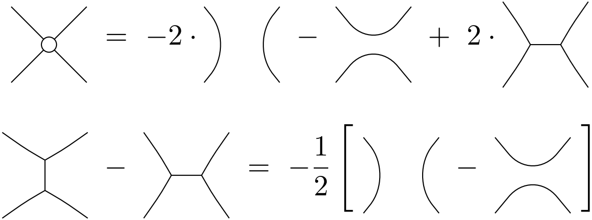
Simple skein theory classification

Why should we care?

- Small objects frequently show up in our calculations, especially in classification questions.
- Having an ability to discern which small object you are working with quickly is very useful.
- In general, classification problems can help us find objects we didn't know existed.

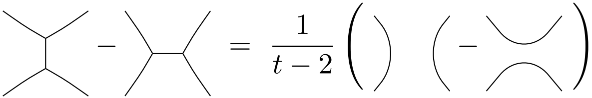
Simple skein theory classification

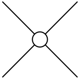
Some of the planar algebras in MPS17 will also be naturally included in this classification. For instance OSp(1|2) has relations:



Simple skein theory classification

- Of course there is no point in doing this classification if nothing new exists!
- One example not included in MPS17 is Deligne's S_t, which has I=H relation:



-  cannot be written as a linear combination of non-virtual diagrams.
- At values of $t \in \mathbb{Z}_{>0}$, Deligne's S_t is a degenerate version of $\mathrm{Rep}(S_t)$.

Classification

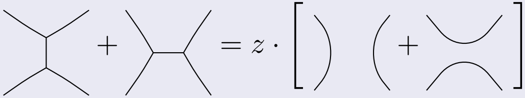
Theorem

Consider $\mathcal{T} = \{T_n\}_{n=0,1,2,\ldots}$, a planar algebra generated by a trivalent vertex and virtual crossing. Any non-fully flat quotient of $\mathcal{T}$ with dimension bounds "1,0,1,1,4,…" is one of the following:

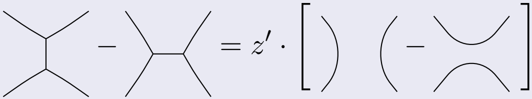
Dimensions	*Name*
$1,0,1,1,3,\ldots$	$Rep(SO(3))$, $Rep(OSp(1\|2))$, *and* $Rep(S_3)$
$1,0,1,1,4,9,\ldots$	$Rep(G_2)$
$1,0,1,1,4,10,\ldots$	$Rep(S_4)$
$1,0,1,1,4,11,\ldots$	S_t *for* $t \notin \mathbb{Z}_{\geq 0}$
$1,0,1,1,4,11,\ldots$	$Rep(S_t)$ *for* $t \in \mathbb{Z}_{\geq 0} - \{0,1,2,3\}$

Lemma

Suppose we have a symmetric trivalent planar algebra with $t \neq 0$ *or* 2. *Then a planar algebra has a relation of the form*

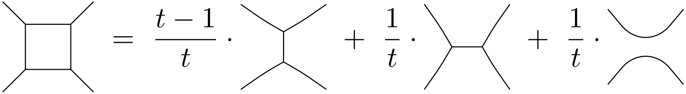


if and only if it has a relation of the form

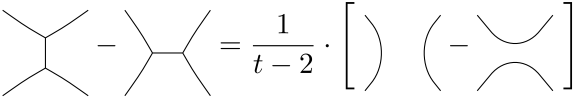


with $z = \dfrac{1}{t}$ *and* $z' = \dfrac{1}{t-2}$.

- Multiplying the top equation by an H we get



- The left side is rotationally invariant, so is the right side!
- Rotating and setting the result equal to each other gives us



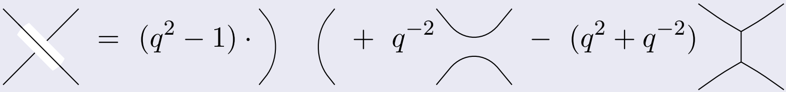
Classification

- Unfortunately (fortunately?), most of the other calculations were done using Mathematica.
- You can download my code from Arxiv and play around with it, but you may have to download the Knot package.
- Essentially, the only symmetric trivalent planar algebras with these conditions are already enumerated in MPS17 or Deligne's S_t.
- The question of which of these categories is braided is also interesting.

Sub-braidings

Theorem

The only non-fully-flat sub-braidings of S_t for generic t and Rep(S_t) for $t \in \mathbb{Z}_{\geq 0} - \{0, 1, 2, 3\}$ is the sub-braiding inherited from $SO(3)_q$:



for $t = q^2 + 2 + q^{-2}$. At $t = 0$ and 3 all sub-braidings are fully-flat.

- This is not a true braiding on the category—no naturality condition with the virtual crossing.
- There is a map from $SO(3)_q \hookrightarrow \text{Rep}(S_t)$ when $t = q^2 + 2 + q^{-2}$.
- Interesting that there are no other sub-braidings.

Future Work

- Repeating this work in characteristic p.
- Working with directed graphs.
- Both are great projects for undergraduates!

Thank you!

Thank you for listening!