Finite Semisimple 2-Categories & Fusion 2-Categories

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Fusion 1-Categories

Why do we care so much?

1. Fusion categories are a generalization of finite groups (or more precisely of their categories of representations).
2. Structured fusion categories give rise to state-sum invariants of 3-manifolds.
3. They are fully dualizable objects of a symmetric monoidal 3-category, and so give 3-dimensional TQFT's.
4. Fusion categories can be constructed from subfactors.
5. They also appear naturally in conformal field theory, when studying representations of quantum groups at roots of unity, in relation to topological orders in 2+1 dimensions, etc.

Fusion 2-Categories

Why should you care?

1. Fusion 2-categories are a generalization of finite 2-groups.
2. Braided fusion categories naturally give rise to fusion 2-categories.
3. Highly structured fusion 2-categories give rise to a state-sum invariant of 4-manifolds.
4. Conjecturally, they are fully dualizable objects of a symmetric monoidal 4-category, and so give 4-dimensional TQFT's.
5. Topological orders in 3+1 dimensions are related to fusion 2-categories.

Module categories over a fusion category

Let us work over a fixed algebraically closed field $\Bbbk$ of characteristic zero, and let $\mathcal{C}$ be a multifusion category.

Definition

A left $\mathcal{C}$-module category is a finite semisimple category $\mathcal{M}$ together with a unital and associative left action of $\mathcal{C}$:

$$\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}.$$

Observation

Left $\mathcal{C}$-module categories, left $\mathcal{C}$-module functors, and left $\mathcal{C}$-module natural transformations form a $\Bbbk$-linear 2-category, which will be denoted by $\mathit{Mod}(\mathcal{C})$.

Examples

$\mathit{Mod}(\mathrm{Vect}_{\mathbb{Z}/p})$

There are two equivalence classes of indecomposable left $\mathrm{Vect}_{\mathbb{Z}/p}$-module categories. They are represented by

$$I := \mathrm{Vect}_{\mathbb{Z}/p} \qquad X := \mathrm{Vect}.$$

The *hom*-categories between these two indecomposable module categories are depicted below.

$$\begin{tikzcd} I \arrow[loop left, "\mathrm{Vect}_{\mathbb{Z}/p\mathbb{Z}}"] \arrow[rr, bend left, "\mathrm{Vect}"] & & X \arrow[ll, bend left, "\mathrm{Vect}"] \arrow[loop right, "\mathrm{Vect}_{\mathbb{Z}/p\mathbb{Z}}"] \end{tikzcd}$$

Remark

There can be non-trivial 1-morphisms between two non-equivalent indecomposable module categories!

Properties of $Mod(\mathcal{C})$

1. Given two left $\mathcal{C}$-module categories $\mathcal{M}, \mathcal{N}$, $Hom(\mathcal{M}, \mathcal{N})$ is a finite semisimple category.
2. Let $F : \mathcal{M} \to \mathcal{N}$ be a left $\mathcal{C}$-module functor. Then, the left and right adjoints of F are left $\mathcal{C}$-module functors.

Definition/Lemma

An left $\mathcal{C}$-module category $\mathcal{M}$ is *simple* if $Id_{\mathcal{M}}$ is a simple object of $End_{Mod(\mathcal{C})}(\mathcal{M})$. In fact, $\mathcal{M}$ is simple if and only if it is indecomposable.

3. Every left $\mathcal{C}$-module category is a direct sum of simple left $\mathcal{C}$-module categories.
4. It has finitely many equivalence classes of simple objects.

Definition [Douglas-Reutter]

A finite presemisimple 2-category is a 2-category satisfying the properties above.

Observations

Remark

A 2-category $\mathfrak{C}$ satisfies 1 and 2 if and only if for every object M of $\mathfrak{C}$, $End_{\mathfrak{C}}(M)$ is a multifusion category.

Example

The 2-category $Mod(\mathcal{C})$ is a finite presemisimple 2-category. Writing $\mathcal{C} = \oplus_{i,j=1}^{n} \mathcal{C}_{ij}$ with the $\mathcal{C}_{ii}$'s fusion categories, let $U\mathcal{C}$ be the 2-category with objects $\{1, ..., n\}$ and $Hom_{U\mathcal{C}}(i, j) = \mathcal{C}_{ij}$. This is a presemisimple 2-category.

Ostrik's Theorem & Separability [EGNO]

Left $\mathcal{C}$-module categories correpond to separable algebras in $\mathcal{C}$.

Consequence

The 2-category $Mod(\mathcal{C})$ is determined to a large extent by $U\mathcal{C}$, which is the full sub-2-category on the object $\mathcal{C}$ as $End_{Mod(\mathcal{C})}(\mathcal{C}) \simeq \mathcal{C}$. How to formalize this?

2-Categorical Interlude I

Fix $\mathfrak{C}$ a locally idempotent complete $\Bbbk$-linear 2-category.

Definition [Gaiotto-Johnson-Freyd]

A 2-condensation in $\mathfrak{C}$ consists of two objects X, and Y together with two 1-morphisms $f : X \to Y$, and $g : Y \to X$ and two 2-morphisms $\phi : f \circ g \Rightarrow Id_Y$ and $\gamma : Id_Y \Rightarrow f \circ g$ such that $\phi \cdot \gamma = Id_{Id_Y}$.

Definition [GJF]

A 2-condensation monad on X in $\mathfrak{C}$ consists of a 1-morphism $e : X \to X$ together with 2-morphisms $\mu : e \circ e \Rightarrow e$ and $\delta : e \Rightarrow e \circ e$ such that μ is associative, δ is coassociative, the Frobenius relations hold, and $\mu \circ \delta = Id_e$.

Definition [GJF]

A 2-condensation monad splits if it can be extended to a 2-condensation.

2-Categorical Interlude II

Definition [GJF]

The 2-category $\mathfrak{C}$ is condensation complete if every 2-condensation monad in $\mathfrak{C}$ splits. It is Cauchy complete if it is condensation complete and has direct sums.

Proposition [GJF, Douglas-Reutter]

There is a natural Cauchy complete 2-category $\mathit{Cau}(\mathfrak{C})$, and a fully faithful inclusion

$$\mathfrak{C} \hookrightarrow \mathit{Cau}(\mathfrak{C})$$

called a Cauchy completion. Further, if $\mathfrak{C}$ is Cauchy complete, this is an equivalence of 2-categories.

Idea behind the construction

The 2-category $\mathit{Cau}(\mathfrak{C})$ is taken to be the 2-category of formal direct sums of condensation monads in $\mathfrak{C}$.

2-Condensation Monads in $\mathit{Mod}(\mathcal{C})$

Proposition [DR]

The canonical inclusion $U\mathcal{C} \hookrightarrow \mathit{Mod}(\mathcal{C})$ is a Cauchy completion.

proof (sketch)

It can be shown that every formal direct sum of 2-condensation monads in $U\mathcal{C}$ is equivalent to a separable algebra in $\mathcal{C}$. So, by the construction of the Cauchy completion, it is enough to prove that $\mathit{Mod}(\mathcal{C})$ is equivalent to the 2-category of separable algebras in $\mathcal{C}$. This follows from Ostrik's theorem and separability.

Corollary

The 2-category $\mathit{Mod}(\mathcal{C})$ is condensation complete.

Finite Semisimple 2-Categories

Definition [Douglas-Reutter]

A finite semisimple 2-category is a $\Bbbk$-linear 2-category that:

1. is locally finite semisimple;
2. has adjoints for 1-morphisms;
3. has direct sums and every 2-condensation monad splits (it is Cauchy complete);
4. has finitely many equivalence classes of simple objects.

Lemma [DR]

Every finite semisimple 2-category is a finite presemisimple 2-category.

Example

We have seen that the 2-category $Mod(\mathcal{C})$ is a finite semisimple 2-category.

Comparison of 3-Categories

Theorem [Douglas-Reutter]

Finite semisimple 2-categories are precisely the 2-categories $Mod(\mathcal{C})$ for some multifusion 2-category $\mathcal{C}$.

proof (sketch)

Let $\mathfrak{C}$ be a finite semisimple 2-category. Let G be the direct sum of the simple objects of $\mathfrak{C}$, and $\mathcal{D} := End_{\mathfrak{C}}(G)$ its endomorphism multifusion category. Then, one can check that $Mod(\mathcal{D}) \simeq \mathfrak{C}$.

Theorem [D]

There is an equivalence of 3-categories:

$$\left\{ \begin{matrix} Multifusion\ \ Categories \\ Bimodules \\ Bimodule\ \ Functors \\ Bimodule\ \ Natural\ \ Transformations \end{matrix} \right\} \to \left\{ \begin{matrix} Finite\ \ Semisimple\ 2 - Categories \\ 2 - Functors \\ 2 - Natural\ \ Transformations \\ Modifications \end{matrix} \right\}$$

Multifusion 2-Categories

Definition [Douglas-Reutter, D]

A multifusion 2-category is a rigid monoidal finite semisimple 2-category. A multifusion 2-category with simple monoidal unit is called a fusion 2-category.

Example

Let $\mathcal{C}$ be a braided fusion category. Then, $Mod(\mathcal{C})$ is a fusion 2-category with monoidal product $\boxtimes_{\mathcal{C}}$.

Fusion in $Mod(\mathrm{Vect}_{\mathbb{Z}/p})$

When we equip $\mathrm{Vect}_{\mathbb{Z}/p}$ with the trivial braiding, we find:
$X \boxtimes_{\mathrm{Vect}_{\mathbb{Z}/p}} X \simeq pX$.
When we equip $\mathrm{Vect}_{\mathbb{Z}/p}$ with a non-trivial braiding, we find:
$X \boxtimes_{\mathrm{Vect}_{\mathbb{Z}/p}} X \simeq I$.

Connected Fusion 2-Categories

Definition

A finite semisimple 2-category $\mathfrak{C}$ is connected if for every simple objects M, N, $Hom_{\mathfrak{C}}(M, N)$ is non-trivial.

Proposition [Douglas-Reutter, D]

Every connected fusion 2-category is of the form $Mod(\mathcal{D})$ for some braided fusion category $\mathcal{D}$.

proof (sketch)

If I denotes the monoidal unit of the connected fusion 2-category $\mathfrak{C}$, then $\mathcal{D} := End_{\mathfrak{C}}(I)$ is a braided fusion category, and one can check that $\mathfrak{C} \simeq Mod(\mathcal{D})$. Given a braided fusion category $\mathcal{D}$, $\mathcal{D}$ viewed as a left $\mathcal{D}$-module category is the monoidal unit of $Mod(\mathcal{D})$. We have $End_{Mod(\mathcal{D})}(\mathcal{D}) \simeq \mathcal{D}$ as braided fusion categories.

More Examples!

Example

Let G be a finite group, then $2\mathrm{Vect}_G$ the category of G-graded 2-vector spaces is a fusion 2-category. Given a $\omega \in H^4(G; \mathbb{C}^\times)$, we can form $2\mathrm{Vect}_G^\omega$ by twisting the associator.

Example [D]

Given $\mathfrak{C}$ and $\mathfrak{D}$ two multifusion 2-categories, one can form their 2-Deligne tensor product $\mathfrak{C} \boxdot \mathfrak{D}$.

Example [Kong-Tian-Zhang, Johnson-Freyd, D]

The Drinfeld center of $2\mathrm{Vect}_G^\omega$ is a fusion 2-category.

Remark

This last example is particularly interesting because these fusion 2-categories are very different from the other examples we considered!

Conclusion

The theory of fusion 2-categories has a lot of similarities with that of fusion categories, but also some striking differences; Most notably, there can exist non-trivial 1-morphisms between non-equivalent simple objects of a given fusion 2-category.

This motivates trying to prove analogues of some classical results in the theory of fusion categories. For instance, one can prove that the 2-Deligne tensor product exist, and that a 2-Ostrik theorem hold. (These constructions will be the subjects of futur papers.)

Another interesting problem is to try to find exotic examples. More precisely, do there exist examples that cannot be built using the examples and techniques touched upon in this talk?