Oct. 24, 2013

Name Key
TA Name ________
Recitation Time ________

Math 2173 – Homework Set 4
Due : Thursday, Oct. 31, In Recitation

1 . (8 points) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

(a) $\mathbf{F}(x, y, z) = \langle z, y, -x \rangle$
$\mathbf{r}(t) = \langle t, \sin t, \cos t \rangle, \ 0 \le t \le \pi$

$$\vec{r}\,'(t) = \langle 1, \cos(t), -\sin(t) \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{\pi} \langle \cos(t), \sin(t), -t \rangle \cdot \langle 1, \cos(t), -\sin(t) \rangle dt$$

$$= \int_0^{\pi} \cos(t) \cdot 1 + \sin(t) \cdot \cos(t) - t \cdot (-\sin(t))\,dt$$

$$= \int_0^{\pi} \cos(t)\,dt + \int_0^{\pi} \sin(t)\cos(t)\,dt + \int_0^{\pi} t\sin(t)\,dt$$

$u = \sin(t)$, $du = \cos(t)dt$

$u = t$, $dV = \sin(t)dt$, $du = dt$, $V = -\cos(t)$

$$= \sin(t)\Big]_{t=0}^{\pi} + \int_0^0 u\,du + \left(-t\cos(t)\right)\Big|_{t=0}^{\pi} - \int_0^{\pi} -\cos(t)\,dt$$

$$= 0 - 0 + 0 - \pi\cos(\pi) + 0 + \left(\sin(t)\right)\Big|_{t=0}^{\pi}$$

$$= \pi$$

(b) $\mathbf{F}(x,y) = \left\langle \dfrac{y^2}{1+x^2}, 2y\arctan x \right\rangle$
$\mathbf{r}(t) = \langle t^2, 2t\rangle,\ 0 \le t \le 1$

$$\vec{r}\,'(t) = \langle 2t, 2\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left\langle \frac{(2t)^2}{1+(t^2)^2}, 2\cdot 2t \arctan(t^2)\right\rangle \cdot \langle 2t, 2\rangle\, dt$$

$$= \int_0^1 \frac{4t^2}{1+t^4}\cdot 2t + 4t\arctan(t^2)\cdot 2\, dt$$

$$= \int_0^1 \frac{8t^3}{1+t^4}\, dt + \int_0^1 8t\arctan(t^2)\, dt$$

$u = 1+t^4$
$du = 4t^3\,dt$

$u = t^2$
$du = 2t\,dt$

$$= \int_1^2 \frac{2\,du}{u} + \int_0^1 4\arctan(u)\,du$$

$P = 4\arctan(u)$ $\quad dQ = du$
$dP = \frac{4}{1+u^2}du$ $\quad Q = u$

$$= 2\ln u\Big|_{u=1}^{2} + 4u\arctan(u)\Big|_0^1 - \int_0^1 \frac{4u}{1+u^2}\,du$$

$V = 1+u^2$
$dV = 2u\,du$

$$2(\ln 2 - \ln 1) + 4\arctan(1) - 0 - \int_1^2 \frac{2}{V}\,dV$$

$$= 2\ln 2 + 4\cdot\frac{\pi}{4} + (-2\ln V)\Big|_{V=1}^{2} =$$

$$= 2\ln 2 + \pi - 2\ln 2 + 2\ln 1 = \pi$$

2 . (7 points) Consider the following initial value problem.

$$y'' + 4y = 5\cos 3t; \;\; y(0) = 4, y'(0) = 2$$

(a) Verify that the functions $y = \sin 2t$ and $y = \cos 2t$ are solutions of the homogeneous equation.

$y = \sin(2t)$

$y' = 2\cos(2t)$

$y'' = -4\sin(2t)$

$y'' + 4y = -4\sin(2t) + 4\sin(2t)$

$= 0.$

$y = \cos(2t)$

$y' = -2\sin(2t)$

$y'' = -4\cos(2t)$

$y'' + 4y = -4\cos(2t) + 4\cos(2t)$

$= 0.$

(b) Verify that the function $y = -\cos 3t$ is a particular solution of the differential equation.

$y = -\cos(3t)$

$y' = 3\sin(3t)$

$y'' = 9\cos(3t)$

$y'' + 4y = 9\cos(3t) + 4(-\cos(3t))$

$= 5\cos(3t)$

(c) Solve the initial value problem.

The general solution is

$y = c_1 \sin(2t) + c_2 \cos(2t) - \cos 3t.$

Initial conditions:

$y(0) = 4$

$y'(0) = 2$

$y(0) = c_1 \cdot \sin(0) + c_2 \cos(0) - \cos(0)$

$= c_1 \cdot 0 + c_2 \cdot 1 - 1 = c_2 - 1 = 4$

$\Rightarrow c_2 = 5$

$y' = 2c_1 \cos(2t) - 2c_2 \sin(2t) + 3\sin 3t$

$y'(0) = 2c_1 \cdot 1 - 2c_2 \cdot 0 + 0 = 2c_1 = 2$

$\Rightarrow c_1 = 1$

So solution to IVP is

$y = \sin(2t) + 5\cos(2t) - \cos(3t)$