Sep. 12, 2013

Name KEY
TA Name
Recitation Time

Math 2173 – Homework Set 2
Due : Thursday, Sep. 19, In Recitation

1 . (4 points) Find the maximum and minimum values of f subject to the given constraint.

$$f(x, y, z) = x^2 + 2y \text{ subject to } x^2 + 2y^2 + z^2 = 16$$

$g = x^2 + 2y^2 + z^2 = 16$

$\nabla f = \langle 2x, 2, 0 \rangle$

$\nabla g = \langle 2x, 4\cancel{x}y, 2z \rangle$

$\boxed{\nabla f = \lambda \nabla g, g = 16}$

① $2x = \lambda \cdot 2x$

② $2 = \lambda \cdot 4y$

③ $0 = \lambda \cdot 2z$

④ $x^2 + 2y^2 + z^2 = 16$

② $\Rightarrow \lambda \neq 0$ and $y \neq 0$

so ③ $\Rightarrow z = 0$

① $\Rightarrow x = \lambda \cdot x$

Case 1. $x = 0$

④ $\Rightarrow 0^2 + 2y^2 + 0^2 = 16$

$y^2 = 8$

$y = \pm\sqrt{8}$

$(0, \pm\sqrt{8}, 0)$

Case 2. $x \neq 0$

$\Rightarrow \lambda = 1$

② $\Rightarrow 2 = 4y \Rightarrow y = \frac{1}{2}$

④ $\Rightarrow x^2 + 2\left(\frac{1}{2}\right)^2 + 0^2 = 16$

$\Rightarrow x^2 = 16 - \frac{1}{2} = \frac{31}{2}$

$x = \pm\sqrt{\frac{31}{2}}$

$\left(\pm\sqrt{\frac{31}{2}}, \frac{1}{2}, 0\right)$

$f(0, \sqrt{8}, 0) = 0^2 + 2\sqrt{8} = 2\sqrt{8}$

$f(0, -\sqrt{8}, 0) = 0^2 - 2\sqrt{8} = -2\sqrt{8}$

$f\left(\pm\sqrt{\frac{31}{2}}, \frac{1}{2}, 0\right) = \frac{31}{2} + \frac{2}{2} = \frac{33}{2}$

(Note: $\frac{33}{2} > 2\sqrt{8}$)

Max is $\frac{33}{2}$ which occurs at points $\left(\pm\sqrt{\frac{31}{2}}, \frac{1}{2}, 0\right)$

Min is $-2\sqrt{8}$ which occurs at point $(0, -\sqrt{8}, 0)$

2 . (4 points) Evaluate $\iint_R \underbrace{3x^2ye^{x^3y}} \, dA$, where $R = \{(x,y) | 0 \le x \le 1,\ 0 \le y \le 2\}$.

if we try to integrate y first we would need to do integration by parts. Let's try to integrate x first.

$$\iint_R 3x^2ye^{x^3y}\,dA = \int_0^2 \underbrace{\int_0^1 3x^2y\,e^{x^3y}\,dx}\,dy$$

$$\int_0^1 3x^2y\,e^{x^3y}\,dx$$

$u = x^3y$

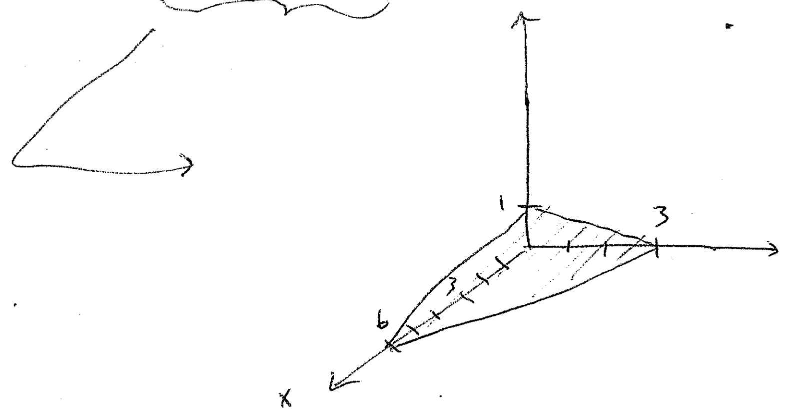
$du = 3x^2y\,dx$ Treat y as a constant

$$= \int e^u\,du = e^u = e^{x^3y}\Big|_{x=0}^{1} = e^{1\cdot y} - e^{0\,y} = e^y - 1$$

$$\Rightarrow \iint_R 3x^2y\,e^{x^3y}\,dA = \int_0^2 e^y - 1\,dy = e^y - y\Big|_{y=0}^{2} = e^2 - 2 - (e^0 - 0)$$

$$= e^2 - 3$$

3 . (3 points) Find the volume of the tetrahedron bounded by the coordinate planes ($x = 0, y = 0, z = 0$) and the plane $x + 2y + 6z = 6$.



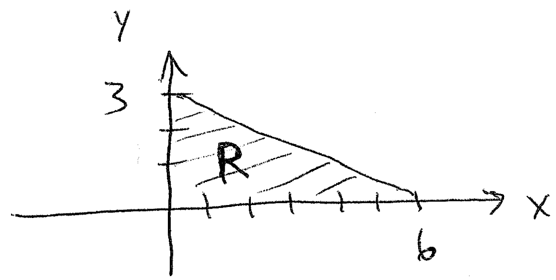
Intersects z-axis when $x=y=0$
$\Rightarrow 6z = 6 \Rightarrow z = 1$
Intersects x-axis when $y=z=0$
$\Rightarrow x = 6$
Intersects y-axis when $x=z=0$
$2y = 6 \Rightarrow y = 3$

$$x + 2y + 6z = 6 \Rightarrow 6z = 6 - x - 2y$$
$$z = 1 - \frac{x}{6} - \frac{y}{3}$$

$$\text{Volume} = \iint_R \text{height } dA = \iint_R 1 - \frac{x}{6} - \frac{y}{3}\, dA$$

From above we see that R is a triangle in x-y plane. Specifically, we consider $z = 0$ for $x + 2y + 6z = 6$.

$$\Rightarrow x + 2y = 6 \Rightarrow 2y = 6 - x \Rightarrow y = 3 - \frac{x}{2}$$



So R is described by
$0 \le x \le 6$
$0 \le y \le 3 - \frac{x}{2}$

$$\text{Volume} = \int_0^6 \int_0^{3-\frac{x}{2}} 1 - \frac{x}{6} - \frac{y}{3}\, dy\, dx = \int_0^6 y - \frac{xy}{6} - \frac{y^2}{6} \Big|_0^{3-\frac{x}{2}} dx$$

$$= \int_0^6 3 - \frac{x}{2} - \frac{x}{6}\left(3 - \frac{x}{2}\right) - \frac{1}{6}\left(3 - \frac{x}{2}\right)^2 dx = \int_0^6 3 - \frac{x}{2} - \frac{x}{2} + \frac{x^2}{12} - \frac{1}{6}\left(9 - 3x + \frac{x^2}{4}\right) dx$$

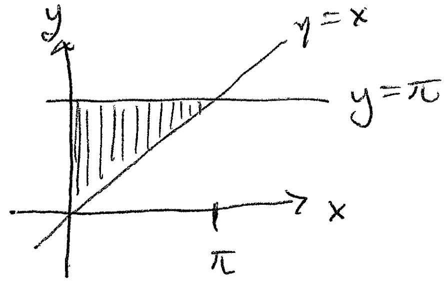
$$= \int_0^6 3 - \frac{x}{2} - \frac{x}{2} + \frac{x^2}{12} - \frac{3}{2} + \frac{x}{2} - \frac{x^2}{24}\, dx = \int_0^6 \frac{3}{2} - \frac{x}{2} + \frac{x^2}{24}\, dx = \frac{3}{2}x - \frac{x^2}{4} + \frac{x^3}{72} \Big|_0^6$$

$$= \frac{3}{2} \cdot 6 - \frac{36}{4} + \frac{216}{72} = 9 - 9 + 3 = 3$$

4 . (4 points) Evaluate $\int_0^\pi \int_x^\pi \cos(y^2)\, dy\, dx$.

As written, we can not evaluate this integral with respect to y first. Let's switch the order of integration.

$0 \le x \le \pi$
$x \le y \le \pi$ $\Rightarrow$



$\hookrightarrow$ written the other way, we see that y varies from 0 to π and then x is between $x = 0$ and $x = y$.

i.e. $0 \le y \le \pi$
$0 \le x \le y$

So $$\int_0^\pi \int_x^\pi \cos(y^2)\, dy\, dx = \int_0^\pi \int_0^y \cos(y^2)\, dx\, dy = \int_0^\pi x \cos(y^2) \Big|_{x=0}^{y} dy$$

$$= \int_0^\pi y \cos(y^2)\, dy$$

$u = y^2$
$du = 2y\, dy$

$$= \frac{1}{2} \int \cos(u)\, du = \frac{\sin(u)}{2}$$

$$= \frac{\sin(y^2)}{2} \Big|_{y=0}^{\pi} = \frac{\sin(\pi^2)}{2} - \frac{\sin(0)}{2} = \frac{\sin(\pi^2)}{2}$$