Aug. 29, 2013

Name KEY
TA Name __________
Recitation Time __________

Math 2173 – Homework Set 1
Due : Thursday, Sep. 5, In Recitation

1 . Consider the following function and point P.

$$f(x,y) = 2\overset{\sin}{\cancel{\cos}}(3x-2y),\ \ P(0,\pi)$$

(a) Find the unit vectors that give the direction of steepest ascent and steepest descent at P.

$\nabla f = \langle 2\cos(3x-2y)\cdot 3,\ 2\cos(3x-2y)\cdot(-2)\rangle$

$\nabla f(0,\pi) = \langle 2\cos(-2\pi)\cdot 3,\ 2\cos(-2\pi)\cdot(-2)\rangle = \langle 6,-4\rangle$

$|\nabla f(0,\pi)| = \sqrt{6^2+4^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

Steepest ascent direction is $\frac{1}{2\sqrt{13}}\langle 6,-4\rangle$ or $\langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\rangle$. Steepest descent is $\langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle$

(b) Find a vector that points in a direction of no change in the function at P.

Any vector perpendicular to $\langle 6,-4\rangle$ works.

Note that $\langle 6,-4\rangle \cdot \langle 4,6\rangle = 6\cdot 4 - 4\cdot 6 = 0$.

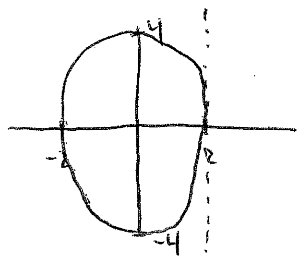
So a vector in direction of no change is $\langle 4,6\rangle$. (Many others)

2 . Consider the paraboloid $f(x,y) = -4 + x^2 + \frac{y^2}{4}$ and the point P on the given level curve of f.

$$f(x,y) = 0,\ \ P(2,0)$$

Compute the slope of the line tangent to the level curve at P and verify that the tangent line is orthogonal to the gradient at that point.

$-4 + x^2 + \frac{y^2}{4} = 0 \Rightarrow \frac{y^2}{4} = 4 - x^2 \Rightarrow y = \pm 2\sqrt{4-x^2}$



slope of tangent line at $(2,0)$ is infinite.

tangent line is $x = 2$

A vector in direction of tangent line is $\langle 0,1\rangle$.

$\nabla f = \langle 2x, \frac{y}{2}\rangle \qquad \nabla f(2,0) = \langle 4,0\rangle$

$\langle 4,0\rangle \cdot \langle 0,1\rangle = 4\cdot 0 + 0\cdot 1 = 0$

So tangent line is orthogonal to gradient at $P(2,0)$.

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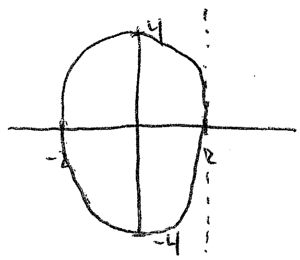

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$\langle 4,0\rangle \cdot \langle 0,1\rangle = 4\cdot 0 + 0\cdot 1 = 0$

So tangent line is orthogonal to gradient at $P(2,0)$.

3 . Find the critical points of the following function. Determine whether each critical point corresponds to a local max, local min, or saddle point.

$$f(x, y) = y^2 + x^2y - 2y + 3$$

$f_x = 2xy$

$f_y = 2y + x^2 - 2$

P.'s when $f_x = 0$ and $f_y = 0$

$x = 0$ or $y = 0$ $\quad$ $2y + x^2 - 2 = 0$

If $x = 0$, then $2y + x^2 - 2 = 0 \Rightarrow 2y - 2 = 0 \Rightarrow y = 1$

So $(0, 1)$ is a critical point.

If $y = 0$, then $2y + x^2 - 2 = 0 \Rightarrow x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$

So $(\pm\sqrt{2}, 0)$ is a critical point.

$f_{xx} = 2y$, $f_{xy} = 2x$, $f_{yy} = 2$

$D = f_{xx} f_{yy} - f_{xy}^2 = 4y - (2x)^2$

At $(0,1)$, $D = 4 > 0$ and $f_{xx} = 2 > 0$.

So local min at $(0,1)$.

At $(\sqrt{2}, 0)$, $D = 0 - (2\sqrt{2})^2 = -8 < 0$

Saddle point at $(\sqrt{2}, 0)$

At $(-\sqrt{2}, 0)$, $D = 0 - (-2\sqrt{2})^2 = -8 < 0$

Saddle point at $(-\sqrt{2}, 0)$

4 . Find the absolute maximum and minimum values of the following function on the given set R.

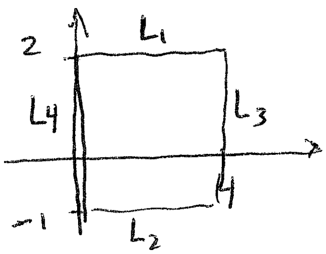
$$f(x,y) = x^2 - 6x + 2y^2 - 1, \;\; R = \{(x,y) | 0 \le x \le 4, -1 \le y \le 2\}$$

For interior, we use critical points.

$f_x = 2x - 6 = 0 \Rightarrow x = 3$ $\qquad$ $f_y = 4y = 0 \Rightarrow y = 0$.

So only critical point is $(3,0)$ [which is inside R].

$f(3,0) = 9 - 18 + 0 - 1 = -10$



Now we find max/min on the lines.

L_1: $y=2$ $\quad 0 \le x \le 4$, $\quad g(x) = f(x,2) = x^2 - 6x + 2\cdot 2^2 - 1 = x^2 - 6x + 7$

$g'(x) = 2x - 6 = 0 \Rightarrow x = 3.$ $\quad g(0) = 7$, $g(3) = -2$, $g(4) = -1$

L_2: $y = -1$, $0 \le x \le 4$, $\quad g(x) = f(x,-1) = x^2 - 6x + 2(-1)^2 - 1 = x^2 - 6x + 1$

$g'(x) = 2x - 6 = 0 \Rightarrow x = 3.$ $\quad g(0) = 1$, $g(3) = -8$, $g(4) = -7$

L_3: $x = 4$, $-1 \le y \le 2$, $\quad h(y) = f(4,y) = 16 - 24 + 2y^2 - 1 = 2y^2 - 9$

$h'(y) = 4y = 0 \Rightarrow y = 0.$ $\quad h(-1) = -7$, $h(0) = -9$, $h(2) = -1$

L_4: $x = 0$, $-1 \le y \le 2$, $\quad h(y) = f(0,y) = 2y^2 - 1$

$h'(y) = 4y = 0 \Rightarrow y = 0.$ $\quad h(-1) = 1$, $h(0) = -1$, $h(2) = 7$

So absolute minimum is -10 which occurs at $(3,0)$.

Absolute maximum is 7 which occurs at $(0,2)$.

5 . Find the maximum and minimum values of f subject to the following constraint.

$$f(x,y) = x^2 + y^2 \text{ subject to } \underbrace{x^4 + y^4}_{g(x,y)} = 1$$

By method of Lagrange multipliers, we want to solve

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g = 1.$$

$$\nabla f = \langle 2x, 2y \rangle$$

$$\nabla g = \langle 4x^3, 4y^3 \rangle$$

① $2x = \lambda \cdot 4x^3$

② $2y = \lambda \cdot 4y^3$

③ $x^4 + y^4 = 1$

If x and y are not zero, then ① and ② become.

$1 = \lambda \cdot 2x^2$, $1 = \lambda \cdot 2y^2$. So λ can't be zero.

$\lambda 2x^2 = \lambda 2y^2 \Rightarrow x^2 = y^2$ (uses) or $y = \pm x$

So ③ becomes $x^4 + (x^2)^2 = 1 \Rightarrow 2x^4 = 1$

$x^4 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt[4]{2}}$ $y = \pm \frac{1}{\sqrt[4]{2}}$.

We have the 4 points $\left(\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}\right), \left(\frac{1}{\sqrt[4]{2}}, \frac{-1}{\sqrt[4]{2}}\right), \left(\frac{-1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}\right), \left(\frac{-1}{\sqrt[4]{2}}, \frac{-1}{\sqrt[4]{2}}\right)$

If $x = 0$, then by ③, $y^4 = 1 \Rightarrow y = \pm 1$ so $(0, \pm 1)$

If $y = 0$, then by ③, $x^4 = 1 \Rightarrow x = \pm 1$ $(\pm 1, 0)$

$$f\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right) = \left(\pm \frac{1}{\sqrt[4]{2}}\right)^2 + \left(\pm \frac{1}{\sqrt[4]{2}}\right)^2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$f(0, \pm 1) = 0^2 + (\pm 1)^2 = 1$$

$$f(\pm 1, 0) = (\pm 1)^2 + 0^2 = 1$$

So minimum is 1 which occurs at $(0, \pm 1)$ and $(\pm 1, 0)$.

and maximum is $\sqrt{2}$ which occurs at $\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right)$