The sum of the reciprocal of the random walk

Jon Michel * Robert de Jong†

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Abstract

This paper derives the limit distribution of the rescaled sum of the reciprocal of the positive part of a random walk with continuously distributed innovations, and of the rescaled sum of the reciprocal of the absolute value of a random walk with continuously distributed innovations. It also considers this statistic for the case of a simple random walk, and shows that the limit distribution is different for this case.

1 Introduction

This paper establishes the asymptotic distribution for the rescaled sum of the reciprocal of the positive part and of the absolute value of the random walk. More precisely, it establishes under regularity conditions for a random walk process x_t with i.i.d. increments ε_t starting at $x_0 = 0$ the limit distribution for

$$A_n = \sigma n^{-1/2}(\log(n))^{-1} \sum_{t=1}^{n} x_t^{-1} I(x_t > 0)$$

and

$$B_n = \sigma n^{-1/2}(\log(n))^{-1} \sum_{t=1}^{n} |x_t|^{-1} I(x_t \neq 0)$$

*Department of Economics, Ohio State University, 304 Arps Hall, Columbus, OH 43210, USA, Email michel.82@osu.edu.

†Department of Economics, Ohio State University, 444 Arps Hall, Columbus, OH 43210, USA, email de-jong.8@osu.edu.

where $E\varepsilon_t = 0$ and $\sigma^2 = E\varepsilon_t^2 \in (0, \infty)$. Throughout this paper, we assume that $n \geq 2$ in order to make divisions by $\log(n)$ be well-defined. This result is related to various results in the literature. First, it is well-known that the continuous mapping theorem ensures convergence in distribution of objects of the form $n^{-1}\sum_{t=1}^{n} f(n^{-1/2}x_t)$ for integrated processes x_t such that $n^{-1/2}x_{[rn]}$ satisfies an invariance principle and functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous on $\mathbb{R}$. Results of this type can be extended to classes of functions that are not continuous, such as functions with a finite number of discontinuities; this is undertaken in for example Park and Phillips (1999). For functions $f(.)$ that are locally integrable, de Jong (2004) and Pötscher (2004) show that this type of asymptotics is also still valid. However, the pole at 0 in the reciprocal poses issues that cannot be resolved along these lines.

The results of this paper are also related to convergence in distribution results for objects of the form $a_n \sum_{t=1}^{n} f(x_t)$, where no rescaling of x_t with $n^{-1/2}$ has taken place. Results for objects of this form can be found in Borodin and Ibragimov (1995) and Park and Phillips (1999) for asymptotically homogeneous, periodic and integrable functions, and de Jong (2010) for the exponential function. Qu and de Jong (2012) consider the exponential of the random walk with drift. De Jong and Wang (2005) considered negative powers smaller than -1 of the unit root process and a "clipping device", viz., the values of the unit root process in a neighborhood of zero is removed from the statistic.

The main inspiration however for this paper is Pötscher (2013), who established the order of magnitude $O_p(n^{\alpha/2})$ for objects of the form $\sum_{t=1}^{n} |x_t|^{-\alpha}$ and $\sum_{t=1}^{n} x_t^{-\alpha} I(x_t > 0)$ for $\alpha > 1$, and the order of magnitude $O_p(n^{1/2}\log(n))$ for $\sum_{t=1}^{n} x_t^{-1} I(x_t > 0)$ and $\sum_{t=1}^{n} |x_t|^{-1} I(x_t > 0)$. This paper seeks to establish the limit distribution for the last two statistics for the random walk case.

One important limitation of our result is that for analytical reasons, it can only deal with the case of i.i.d. increments Δx_t, and therefore (unlike the results of Pötscher (2013)) it does not address the more general case of a unit root process with weakly dependent increments Δx_t. However, since our computer simulations indicate that extremely large sample sizes are needed to approximate the limit distributions found in Theorem 1 and 2 (below), such a result would likely not be of much practical value anyway.

2 Assumptions and main results

The maintained assumption in this paper is the following:

Assumption 1. *$x_0 = 0$, and $\Delta x_t = \varepsilon_t$, $t = 1, 2, ...$, where ε_t, $t = 1, 2, ...$ is an i.i.d. sequence of random variables such that $\sigma^2 = E\varepsilon_t^2 > 0$ and $E|\varepsilon_t|^p < \infty$ for some $p > 2$. There exists a $\beta > 1$ such that the characteristic function $\phi(r) = E\exp(ir\varepsilon_t)$ satisfies $\lim_{r\to\infty} r^\beta \phi(r) = 0$. The density $h(.)$ of ε_t possesses a derivative $h'(.)$ that is continuous on $\mathbb{R}$ and satifies*

$\int_{-\infty}^{\infty} \sup_{z\in[0,\eta]} |h'(z-x)|dx < \infty$ *for some* $\eta > 0$. *In addition,* $h(.) \leq R(x)$ *for some function* $R(.)$ *that is integrable, continuous on* $\mathbb{R}$*, and monotone on* $(0,\infty)$ *and* $(-\infty,0)$*.*

The condition $\lim_{r\to\infty} r^{\beta}\phi(r) = 0$ for some $\beta > 1$ is not uncommon in the literature. It implies that the density $f_t(.)$ of $t^{-1/2}x_t$ satisfies $F = \sup_{t\geq 1, y\in\mathbb{R}} f_t(y) < \infty$; see the discussion in Pötscher (2013). It may be possible to obtain results under the condition $\beta > 0$ rather than $\beta > 1$, which implies that $t^{-1/2}x_t$ has a bounded density uniformly over $t \geq M$, for some $M > 0$ following the approach outlined in Pötscher (2013); however, we will not pursue that here. The condition $\int_{-\infty}^{\infty} \sup_{z\in[0,\eta]} |h'(z-x)|dx < \infty$ is nonstandard. This condition on $h'(.)$ will be satisfied if for some nonincreasing function $Q : [0,\infty) \to \mathbb{R}$, $|h'(x)| \leq Q(|x|)$ and $\int_0^{\infty} Q(x)dx < \infty$. For the standard normal density function, this condition holds for $Q(x) = \phi(1)I(0 \leq x \leq 1) + x\phi(x)I(x > 1)$. Under Assumption 1, the conditions of Lemma 1 of page 64 of Akonom (1993) and the conditions of the strong approximation result of Lemma 2.3(b) of page 271 of Park and Phillips (1999) are satisfied, and both results will be used in the proof.

The main result of this note is the following.

Theorem 1. *Under Assumption 1,* $A_n \xrightarrow{d} |Z|$ *and* $B_n \xrightarrow{d} 2|Z|$*, where* $Z \sim N(0,1)$*.*

The proofs of the results of this paper are gathered in the Mathematical Appendix. The strategy of the proof of Theorem 1 is to write, setting $\sigma = 1$ for clarity of exposition,

$$
\begin{aligned}
A_n = {} & n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(0 < x_t < \delta n^{-1/2}) \\
& + n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\delta n^{-1/2} \leq x_t \leq \eta) \\
& + n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\eta < x_t \leq \eta[n^{\phi}]) \\
& + n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\eta[n^{\phi}] < x_t),
\end{aligned} \tag{1}
$$

where η is the positive constant of Assumption 1, δ is chosen arbitrarily small, and ϕ is a constant inside the $(1/p, 1/2)$ interval. For B_n, a similar reasoning applies by writing out the summation into 8 instead of 4 terms. Here and below, we define the indicator function of an empty set to equal 0. Under Assumption 1, the first term is asymptotically unimportant, while the other three terms converge in distribution; the second term converges in distribution to $(1/2)|Z|$, the third to $\phi|Z|$, and the fourth to $(1/2-\phi)|Z|$. The result then follows by observing that the convergence of the three terms is joint and towards $((1/2)|Z|, \phi|Z|, (1/2-\phi)|Z|)$. Also note that the i.i.d. assumption is needed in the proof of convergence in distribution for the second term only.

Remark. The line of proof used here in its current form does not seem to be capable of generating a result for scaled versions of $\sum_{t=1}^{n}|x_t|^{-\alpha}$ or $\sum_{t=1}^{n}x_t^{-\alpha}I(x_t>0)$ for $\alpha>1$. For the case $\alpha=2$ for example, the result of Lemma 2 can be replaced by the inequality $Ex_t^{-4}I(\delta n^{-1/2}\leq x_t)\leq C_\delta t^{-1/2}n^{3/2}$, implying that proof of Lemma 3 will lead, mutatis mutandis, to the observation that

$$E(\sum_{t=2}^{n}y_{nt})^2=O(n^2).$$

Therefore, an analogue of the result of Lemma 3 can only possibly go through if we are seeking to show convergence in distribution for an object $r_n\sum_{t=1}^{n}x_t^{-2}$ if $nr_n\to 0$. However, such an object necessarily converges to 0 in probability, since Pötscher (2013) shows that $n^{-1}\sum_{t=1}^{n}x_t^{-2}=O_p(1)$.

The continuity of the distribution of ε_t is among the maintained assumptions in Assumption 1. The case of discrete distributions of ε_t requires an alternative proof, for simplicity we consider only the case of a simple random walk.

Theorem 2. *Assume that $x_0=0$, $\Delta x_t=\varepsilon_t$, and ε_t, $t=1,...,$ is an i.i.d. sequence of random variables such that $P(\varepsilon_t=1)=P(\varepsilon_t=-1)=1/2$. Then $A_n\xrightarrow{d}(1/2)|Z|$ and $B_n\xrightarrow{d}|Z|$, where $Z\sim N(0,1)$.*

A surprising aspect of Theorem 2 is that the limit distribution is $(1/2)|Z|$ instead of $|Z|$, as was the case of a continuous distribution for ε_t. This appears to happen because the second term in Equation (1) equals 0 a.s. for the simple random walk case, while for the continuous case, this term contributed $(1/2)|Z|$ to the limit distribution.

For the case of simple random walks and integrable functions $f(\cdot)$ the general case is well understood in the literature, see Dobrushin (1955). There it is shown that $n^{-1/2}\sum_{t=1}^{n}f(x_t)\xrightarrow{d}\sum_{k\in\mathbb{Z}}f(k)|Z|$. This result, again, contrasts with the case where ε_t has a continuous distribution since, as in the earlier remark, the limit distribution for $\sum_{t=1}^{n}|x_t|^{-\alpha}$ with $\alpha>1$ is unknown.

References

Akonom, J. (1993), Comportement asymptotique du temps d'occupation du processus des sommes partielles. Annales de l'Institut Henri Poincaré 29, 57-81.

Borodin, A.N. and I.A. Ibragimov (1995), Limit theorems for functionals of random walks, *Proceedings of the Steklov Institute of Mathematics* 195(2).

de Jong, R.M. (2004), Addendum to 'Asymptotics for nonlinear transformations of integrated time series'. *Econometric Theory* 20, 2004, 627-635.

de Jong, R.M. and C. Wang (2005), Further results on the asymptotics for nonlinear transformations of integrated time series, *Econometric Theory* 21, 413-430.

de Jong, R.M. (2010), Exponentials of unit root processes, working paper, Ohio State University.

Dobrushin, R. L. (1955), Two limit theorems for the simplest random walk on a line (in Russian). Uspekhi Mat. Nauk 10(3) (65), 139-146.

Karatzas, I. and S. Shreve (1991), *Brownian motion and stochastic calculus*, second edition. New York: Springer-Verlag.

Park, J.Y. and P.C.B. Phillips (1999), Asymptotics for nonlinear transformations of integrated time Series", *Econometric Theory* 15, 269-298.

Pötscher, B.M. (2004), Nonlinear functions and convergence to Brownian motion: beyond the continuous mapping theorem, *Econometric Theory* 20, 1-22.

Pötscher, B.M. (2013), On the order of magnitude of sums of negative powers of integrated processes, *Econometric Theory* 29, 642-658.

Qu, X. and R.M. de Jong (2012), Sums of exponentials of random walks with drift, *Econometric Theory* 28, 915-924.

Revész, P. (1981), Local time and invariance, Lecture Notes in Mathematics 861, Analytical Methods in Probability Theory, Proceedings, Oberwolfach, Germany, 128-145.

Mathematical Appendix

Everywhere below, it is assumed that $E\varepsilon_t^2 = 1$.

Lemma 1. *If* $F = \sup_{t \geq 1, y \in \mathbb{R}} f_t(y) < \infty$, *then*

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P(n^{-1/2}(\log(n))^{-1} \sum_{t=1}^{n} x_t^{-1} I(0 < x_t < \delta n^{-1/2}) \neq 0) = 0.$$

Proof. This result follows because

$$\begin{aligned}
&\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P(n^{-1/2}(\log(n))^{-1} \sum_{t=1}^{n} x_t^{-1} I(0 < x_t < \delta n^{-1/2}) \neq 0) \\
&\leq \lim_{\delta \downarrow 0} \limsup_{n \to \infty} P(\exists t \in \{1, ..., n\} : 0 < x_t t^{-1/2} \leq \delta n^{-1/2} t^{-1/2}) \\
&\leq \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sum_{t=1}^{n} P(0 < x_t t^{-1/2} \leq \delta n^{-1/2} t^{-1/2}) \\
&\leq \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sum_{t=1}^{n} \int_0^{\delta n^{-1/2} t^{-1/2}} f_t(y) dy \\
&\leq \lim_{\delta \downarrow 0} \limsup_{n \to \infty} F \sum_{t=1}^{n} \delta n^{-1/2} t^{-1/2} = 0.
\end{aligned}$$

□

Lemma 2. *If* $F < \infty$, *then there exists a constant* $C_\delta > 0$ *such that*

$$E|x_t|^{-2} I(\delta n^{-1/2} \leq x_t) \leq C_\delta t^{-1/2} n^{1/2}.$$

Proof. This follows because

$$\begin{aligned}
&E|x_t|^{-2} I(\delta n^{-1/2} \leq x_t) = t^{-1} E|t^{-1/2} x_t|^{-2} I(\delta n^{-1/2} t^{-1/2} \leq t^{-1/2} x_t) \\
&= t^{-1} \int_{\delta n^{-1/2} t^{-1/2}}^{\infty} y^{-2} f_t(y) dy \\
&\leq t^{-1} F \int_{\delta n^{-1/2} t^{-1/2}}^{\infty} y^{-2} dy \\
&= t^{-1} (\delta n^{-1/2} t^{-1/2})^{-1} F \\
&\leq C_\delta t^{-1/2} n^{1/2}.
\end{aligned}$$

□

Lemma 3. *Let $\Omega_t = \sigma(\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t\})$. Then if $F < \infty$,*

$$n^{-1/2}(\log(n))^{-1} \sum_{t=1}^{n} x_t^{-1} I(\delta n^{-1/2} \leq x_t \leq \eta)$$

$$= o_p(1) + n^{-1/2}(\log(n))^{-1} \sum_{t=2}^{n} E(x_t^{-1} I(\delta n^{-1/2} \leq x_t \leq \eta)|\Omega_{t-1}).$$

Proof. The difference between both statistics equals

$$n^{-1/2}(\log(n))^{-1} x_1^{-1} I(\delta n^{-1/2} \leq x_1 \leq \eta) + n^{-1/2}(\log(n))^{-1} \sum_{t=2}^{n} y_{nt}$$

where

$$y_{nt} = x_t^{-1} I(\delta n^{-1/2} \leq x_t \leq \eta) - E(x_t^{-1} I(\delta n^{-1/2} \leq x_t \leq \eta)|\Omega_{t-1}),$$

and this y_{nt} is a martingale difference array. Obviously $n^{-1/2}(\log(n))^{-1} x_1^{-1} I(\delta n^{-1/2} \leq x_1 \leq \eta) = o_p(1)$, while by Lemma 2,

$$E(\sum_{t=2}^{n} y_{nt})^2 = \sum_{t=2}^{n} E y_{nt}^2 \leq \sum_{t=2}^{n} E|x_t|^{-2} I(\delta n^{-1/2} \leq x_t)$$

$$\leq C_\delta \sum_{t=1}^{n} t^{-1/2} n^{1/2} = O(n).$$

The above implies that

$$E(n^{-1/2}(\log(n))^{-1} \sum_{t=2}^{n} y_{nt})^2 = O((\log(n))^{-2}) = o(1),$$

which establishes the result. $\square$

Lemma 4. *If $F < \infty$, $h'(.)$ is continuous on $\mathbb{R}$, and $\int_{-\infty}^{\infty} \sup_{z \in [0,\eta]} |h'(z-x)| dx < \infty$, we have*

$$n^{-1/2}(\log(n))^{-1} \sum_{t=2}^{n} \int_{\delta n^{-1/2}}^{\eta} z^{-1} h(z - x_{t-1}) dz = o_p(1) + (1/2) n^{-1/2} \sum_{t=2}^{n} h(-x_{t-1}).$$

Proof. We have, for all $\eta > 0$,

$$|\int_{\delta n^{-1/2}}^{\eta} z^{-1}(h(z-x) - h(-x))dz|$$

$$\leq \int_{\delta n^{-1/2}}^{\eta} z^{-1} z \sup_{y\in[0,\eta]} |h'(y-x)|dz$$

$$\leq \eta \sup_{y\in[0,\eta]} |h'(y-x)|.$$

Therefore

$$E|n^{-1/2}(\log(n))^{-1}\sum_{t=2}^{n}\int_{\delta n^{-1/2}}^{\eta} z^{-1}h(z-x_{t-1})dz - (\log(n))^{-1}\int_{\delta n^{-1/2}}^{\eta} z^{-1}dz n^{-1/2}\sum_{t=2}^{n} h(-x_{t-1})|$$

$$\leq \eta n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} E \sup_{z\in[0,\eta]} |h'(z-x_t)|$$

$$= \eta n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n}\int_{-\infty}^{\infty} \sup_{z\in[0,\eta]} |h'(z-t^{1/2}x)|f_t(x)dx$$

$$\leq \eta n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} t^{-1/2}F\int_{-\infty}^{\infty} \sup_{z\in[0,\eta]} |h'(z-x)|dx$$

$$= O((\log(n))^{-1})$$

by assumption. Also, $(\log(n))^{-1}\int_{\delta n^{-1/2}}^{\eta} z^{-1}dz \to 1/2$, and this observation completes the proof of the lemma. □

Lemma 5. *Under Assumption 1,*

$$n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\delta n^{-1/2} \leq x_t \leq \eta) = o_p(1) + (1/2)n^{-1/2}\sum_{t=2}^{n} h(-x_{t-1})|.$$

Proof. By Lemma 3, we have

$$n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\delta n^{-1/2} \leq x_t \leq \eta)$$

$$= o_p(1) + n^{-1/2}(\log(n))^{-1}\sum_{t=2}^{n} E(x_t^{-1}I(\delta n^{-1/2} \le x_t \le \eta)|\Omega_{t-1}).$$

Now using the i.i.d. property of ε_t, we have

$$n^{-1/2}(\log(n))^{-1}\sum_{t=2}^{n} E(x_t^{-1}I(\delta n^{-1/2} \le x_t \le \eta)|\Omega_{t-1})$$

$$= n^{-1/2}(\log(n))^{-1}\sum_{t=2}^{n}\int_{-\infty}^{\infty}(y + x_{t-1})^{-1}I(\delta n^{-1/2} \le y + x_{t-1} \le \eta)h(y)dy$$

$$= n^{-1/2}(\log(n))^{-1}\sum_{t=2}^{n}\int_{\delta n^{-1/2}}^{\eta} z^{-1}h(z - x_{t-1})dz.$$

Now by Lemma 4, the result follows. □

Lemma 6. *If $F < \infty$,*

$$n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1}I(\eta \le x_t \le \eta[n^{\phi}])$$

$$= o_p(1) + (\log(n))^{-1}\sum_{j=2}^{[n^{\phi}]}(j\eta)^{-1}n^{-1/2}\sum_{t=1}^{n} I((j-1)\eta \le x_t \le j\eta).$$

Proof. This follows because

$$n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1}I(\eta \le x_t \le \eta[n^{\phi}]) \ge (\log(n))^{-1}\sum_{j=2}^{[n^{\phi}]}(j\eta)^{-1}n^{-1/2}\sum_{t=1}^{n} I((j-1)\eta \le x_t \le j\eta)$$

and

$$n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1}I(\eta \le x_t \le \eta[n^{\phi}]) \le (\log(n))^{-1}\sum_{j=2}^{[n^{\phi}]}((j-1)\eta)^{-1}n^{-1/2}\sum_{t=1}^{n} I((j-1)\eta \le x_t \le j\eta)$$

and the expectation of the absolute value of the difference between the upper and lower bound is bounded by

$$(\log(n))^{-1}\sum_{j=2}^{[n^\phi]}\eta^{-1}(j(j-1))^{-1}n^{-1/2}\sum_{t=1}^{n}EI((j-1)\eta\leq x_t\leq j\eta)$$

$$\leq(\log(n))^{-1}\sum_{j=2}^{[n^\phi]}\eta^{-1}(j(j-1))^{-1}n^{-1/2}\sum_{t=1}^{n}\eta t^{-1/2}F=O((\log(n))^{-1}). \qquad \square$$

Lemma 7. *For $0<\phi<1/2$, under the conditions of Akonom's (1993) Lemma 1 of page 64 (which are fulfilled under assumption 1),*

$$(\log(n))^{-1}\sum_{j=2}^{[n^\phi]}(j\eta)^{-1}n^{-1/2}\sum_{t=1}^{n}I((j-1)\eta\leq x_t\leq j\eta)$$

$$=o_p(1)+(\log(n))^{-1}\sum_{j=2}^{[n^\phi]}(j\eta)^{-1}n^{-1/2}\sum_{t=1}^{n}I(0\leq x_t\leq\eta).$$

Proof. Note that, by the norm inequality,

$$\alpha_n\equiv E|(\log(n))^{-1}\sum_{j=2}^{[n^\phi]}(j\eta)^{-1}n^{-1/2}\sum_{t=1}^{n}I((j-1)\eta\leq x_t\leq j\eta)$$

$$-(\log(n))^{-1}\sum_{j=2}^{[n^\phi]}(j\eta)^{-1}n^{-1/2}\sum_{t=1}^{n}I(0\leq x_t\leq\eta)|$$

$$\leq(\log(n))^{-1}\sum_{j=2}^{[n^\phi]}(j\eta)^{-1}n^{-1/2}(E|\sum_{t=1}^{n}I((j-1)\eta\leq x_t\leq j\eta)-I(0\leq x_t\leq\eta)|^2)^{1/2}$$

and by applying Lemma 1 on page 64 of Akonom (1993), it follows by setting Akonom's (r,a,δ,k) to $(1,0,\eta,j-1)$ that for $\eta\geq n^{-1/2}$ and $j\geq 2$,

$$E|\sum_{t=1}^{n}I((j-1)\eta\leq x_t\leq j\eta)-I(0\leq x_t\leq\eta)|^2\leq C\eta n^{1/2}(1+(j-1)\eta^2\log(n)),$$

and therefore

$$\alpha_n \le (\log(n))^{-1} \sum_{j=2}^{[n^\phi]} (j\eta)^{-1} n^{-1/2} (C\eta n^{1/2}(1+(j-1)\eta^2 \log(n)))^{1/2}$$

$$= O(n^{-1/4}) + O((\log(n))^{-1/2} n^{\phi/2-1/4}) = o(1). \qquad \square$$

Collecting results and observing that for all $\eta > 0$,

$$(\log(n))^{-1} \sum_{j=2}^{[n^\phi]} (j\eta)^{-1} \to \phi\eta^{-1},$$

we now have

Lemma 8. *Under Assumption 1, for any $\phi \in (0, 1/2)$,*

$$n^{-1/2}(\log(n))^{-1} \sum_{t=1}^{n} x_t^{-1} I(\eta \le x_t \le \eta[n^\phi]) = o_p(1) + \phi\eta^{-1} n^{-1/2} \sum_{t=1}^{n} I(0 \le x_t \le \eta).$$

Proof. This now follows directly from Lemma 6 and 7. $\square$

Proof of Theorem 1. It suffices to show the result for A_n. We first note that if $\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P(Y_n \ne X_{n\delta}) = 0$ and $X_{n\delta} \xrightarrow{d} Y$ for all $\delta > 0$, then $Y_n \xrightarrow{d} Y$. This is because

$$\limsup_{n\to\infty} |E\exp(irY_n) - E\exp(irY)|$$

$$= \limsup_{\delta\downarrow 0} \limsup_{n\to\infty} |E\exp(irY_n) - E\exp(irY)|$$

$$\le \limsup_{\delta\downarrow 0} \limsup_{n\to\infty} |E\exp(irY_n) - E\exp(irX_{n\delta})| + \limsup_{\delta\downarrow 0} \limsup_{n\to\infty} |E\exp(irY) - E\exp(irX_{n\delta})|.$$

The second term is 0 because $X_{n\delta} \xrightarrow{d} Y$ by assumption, while

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} |E\exp(irY_n) - E\exp(irX_{n\delta})|$$

$$\le \lim_{\delta\downarrow 0} \limsup_{n\to\infty} P(Y_n \ne X_{n\delta}) = 0$$

by assumption. Write

$$Y_n = n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(x_t > 0)$$

$$= n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(0 < x_t < \delta n^{-1/2})$$

$$+n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\delta n^{-1/2} \le x_t \le \eta)$$

$$+n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\eta < x_t \le \eta[n^{\phi}])$$

$$+n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\eta[n^{\phi}] < x_t),$$

and set

$$X_{n\delta} = Y_n - n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(0 < x_t < \delta n^{-1/2}).$$

Then from Lemma 1 it follows that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P(Y_n \neq X_{n\delta}) = 0,$$

implying that it now suffices to show that for all $\delta > 0$, $X_{n\delta} \xrightarrow{d} Y$. To show this, note that from Lemma 5 and 8, it follows that

$$X_{n\delta} = o_p(1) + (1/2)n^{-1/2}\sum_{t=2}^{n} h(-x_{t-1})$$

$$+\phi\eta^{-1}n^{-1/2}\sum_{t=1}^{n} I(0 \le x_t \le \eta) + n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\eta[n^{\phi}] < x_t).$$

Since $h(\cdot)$ and $\eta^{-1}I(0 \le \cdot \le \eta)$ integrate to 1, it now follows from Theorem 2 of de Jong and Wang (2005) that under Assumption 1, letting $L(t,s)$ denote the Brownian local time as in Park and Phillips (1999),

$$(1/2)n^{-1/2}\sum_{t=2}^{n} h(-x_{t-1}) \xrightarrow{d} (1/2)L(1,0) \tag{2}$$

and

$$\phi\eta^{-1}n^{-1/2}\sum_{t=1}^{n} I(0 \leq x_t \leq \eta) \xrightarrow{d} \phi L(1,0). \tag{3}$$

Note that the second result does not formally follow directly from Theorem 2 of de Jong and Wang (2005) because $I(0 \leq x \leq \eta)$ is not continuous; however, the result can be easily shown to follow by using continuous upper and lower approximations to this function. Defining $W_n^0(r) = n^{-1/2}x_{[rn]}$ for $r \in [0,1]$ it follows from Park and Phillips (1999, Lemma 2.3b) that under Assumption 1, there exist processes $W_n(.)$ and $W(.)$ such that $W_n^0 \stackrel{d}{=} W_n$ and $\sup_{r\in[0,1]} |W_n(r) - W(r)| = o_p(n^{-(p-2)/2p})$. Therefore, noting that for any Borel measurable function $f(.)$ we have

$$\int_0^1 f(W_n^0(r+1/n))dr = \sum_{t=1}^{n} \int_{(t-1)/n}^{t/n} f(W_n^0(r+1/n))dr$$

$$= n^{-1}\sum_{t=1}^{n} f(W_n^0(t/n)) = n^{-1}\sum_{t=1}^{n} f(n^{-1/2}x_t),$$

we have

$$n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(\eta[n^{\phi}] < x_t)$$

$$= (\log(n))^{-1}n^{-1}\sum_{t=1}^{n}(n^{-1/2}x_t)^{-1} I(\eta[n^{\phi}]n^{-1/2} < n^{-1/2}x_t)$$

$$= (\log(n))^{-1}\int_0^1 W_n^0(r+1/n)^{-1} I(\eta[n^{\phi}]n^{-1/2} < W_n^0(r+1/n))dr$$

$$\stackrel{d}{=} (\log(n))^{-1}\int_0^1 W_n(r+1/n)^{-1} I(\eta[n^{\phi}]n^{-1/2} < W_n(r+1/n))dr. \tag{4}$$

Also, since $\sup_{r\in[0,1]} |W_n(r)-W(r)| = o_p(n^{-(p-2)/2p})$ and $\lim_{n\to\infty} n^{(p-2)/2p} \sup_{r\in[0,1-1/n]} |W(r)-W(r+1/n)| = 0$ a.s. by Lévy's global modulus of continuity for Brownian motion (see for example Karatzas and Shreve (1991), p. 114, Theorem 9.25), with probability approaching 1 as $n \to \infty$ we have for every $r \in [0, 1-1/n]$

$$\lim_{n\to\infty} P(W(r) - n^{-(p-2)/(2p)} \leq W_n(r+1/n) \leq W(r) + n^{-(p-2)/(2p)} \ \ \forall r \in [0, 1-1/n]) = 1.$$

Furthermore, note that for all $\eta > 0$, setting $\psi_n = (\log(n))^{-1}$ and choosing $\phi \in (1/p, 1/2)$,

$$(\log(n))^{-1} \int_{\eta[n^\phi]n^{-1/2} - n^{-(p-2)/(2p)}}^{\psi_n} (s - n^{-(p-2)/(2p)})^{-1} ds$$

$$= (\log(n))^{-1} [\log(s - n^{-(p-2)/(2p)})]_{\eta[n^\phi]n^{-1/2} - n^{-(p-2)/(2p)}}^{\psi_n} \longrightarrow 1/2 - \phi,$$

implying that

$$(\log(n))^{-1} \int_{\eta[n^\phi]n^{-1/2} - n^{-(p-2)/(2p)}}^{\psi_n} (s - n^{-(p-2)/(2p)})^{-1} (L(1,s) - L(1,0)) ds$$

$$\leq \sup_{0 \leq s \leq \psi_n} |L(1,s) - L(1,0)| (1/2 - \phi + o_p(1)) = o_p(1)$$

and

$$(\log(n))^{-1} \int_{\psi_n}^{\infty} (s - n^{-(p-2)/(2p)})^{-1} L(1,s) ds$$

$$\leq \sup_{s \in \mathbb{R}} L(1,s) (\log(n))^{-1} \int_{\psi_n}^{\sup_{r \in [0,1]} W(r)} (s - n^{-(p-2)/(2p)})^{-1} ds$$

$$\leq \sup_{s \in \mathbb{R}} L(1,s) (\log(n))^{-1} [\log(s - n^{-(p-2)/(2p)})]_{\psi_n}^{\sup_{r \in [0,1]} W(r)} = o_p(1).$$

Therefore, with probability approaching 1, the term from Equation (4) is bounded from above by

$$(\log(n))^{-1} \int_0^1 (W(r) - n^{-(p-2)/(2p)})^{-1} I(\eta[n^\phi]n^{-1/2} < W(r) + n^{-(p-2)/(2p)}) dr$$

$$= (\log(n))^{-1} \int_{\eta[n^\phi]n^{-1/2} - n^{-(p-2)/(2p)}}^{\infty} (s - n^{-(p-2)/(2p)})^{-1} L(1,s) ds$$

$$= (\log(n))^{-1} L(1,0) \int_{\eta[n^\phi]n^{-1/2} - n^{-(p-2)/(2p)}}^{\psi_n} (s - n^{-(p-2)/(2p)})^{-1} ds$$

$$+ (\log(n))^{-1} \int_{\eta[n^\phi]n^{-1/2} - n^{-(p-2)/(2p)}}^{\psi_n} (s - n^{-(p-2)/(2p)})^{-1} (L(1,s) - L(1,0)) ds$$

$$+ (\log(n))^{-1} \int_{\psi_n}^{\infty} (s - n^{-(p-2)/(2p)})^{-1} L(1,s) ds$$

$$= o_p(1) + (1/2 - \phi)L(1,0),$$

and the same limit result can be obtained for the lower upper bound. Since the proofs of the results of Equation (2) and (3) rest on an application of a strong approximation similar to that used for the above proof (see the proof of Theorem 2 of de Jong and Wang (2005)), the convergence in distribution of the three terms is joint, and to the same random variable $L(1,0)$. Therefore, summing up, we now have $X_{n\delta} \xrightarrow{d} L(1,0)$, and therefore

$$Y_n = n^{-1/2}(\log(n))^{-1}\sum_{t=1}^{n} x_t^{-1} I(x_t > 0) \xrightarrow{d} L(1,0).$$

To conclude the proof, it only remains to show that $L(1,0) \stackrel{d}{=} |Z|$. According to Akonom (1993, page 58), for every t and s, the distribution function of $L(t,s)$ is given by

$$P(L(t,s) \le x) = (2\Phi(\frac{|s|+x}{\sqrt{t}}) - 1)I(x > 0),$$

where $\Phi(.)$ denotes the standardnormal distribution function. This implies that

$$P(L(1,0) \le x) = (2\Phi(x) - 1)I(x > 0),$$

implying that $L(1,0) \stackrel{d}{=} |Z|$ where Z is $N(0,1)$ distributed. □

Proof of Theorem 2. We show the result for A_n; the result for B_n then follows by symmetry. By Proposition 1 of Revész (1981), there exists x_t' and $W(r)$ such that $x_t' \stackrel{d}{=} x_t$, $W(r)$ is standard Brownian motion, and $\sup_{k\in\mathbb{Z}} |n^{-1/2}\sum_{t=1}^n I(x_t' = k) - L(1, k/\sqrt{n})| \xrightarrow{a.s.} 0$, where $L(t,s)$ is the local time of $W(r)$. We note that

$$A_n \stackrel{d}{=} A_n' = \log(n)^{-1} n^{-1/2} \sum_{t=1}^{n} x_t'^{-1} I(x_t' > 0) = \log(n)^{-1} n^{-1/2} \sum_{k=1}^{\infty} k^{-1} \sum_{t=1}^{n} I(x_t' = k). \quad (5)$$

We let $A_{n,\delta}' \equiv \log(n)^{-1} n^{-1/2} \sum_{k=1}^{\delta\sqrt{n}} k^{-1} \sum_{t=1}^{n} I(x_t' = k)$. We observe that

$$P(A_n' \ne A_{n,\delta}') \le P(\max_{1\le t\le n} x_t' > \delta\sqrt{n}) \to P(\max_{r\in[0,1]} W(r) > \delta) \text{ as } n \to \infty. \quad (6)$$

Therefore,

$$\lim_{\delta\to\infty} \limsup_{n\to\infty} P(A_n' \ne A_{n,\delta}') = 0. \quad (7)$$

We will now show that $A'_{n,\delta} \xrightarrow{d} (1/2)|Z|$, for all $\delta > 0$, which will then imply that $A'_n \xrightarrow{d} (1/2)|Z|$ and therefore $A_n \xrightarrow{d} (1/2)|Z|$. To show this, note that

$$|\log(n)^{-1} \sum_{k=1}^{\delta\sqrt{n}} k^{-1}(n^{-1/2} \sum_{t=1}^{n} I(x'_t = k) - L(1, k/\sqrt{n}))| \tag{8}$$

$$\leq \log(n)^{-1} \sum_{k=1}^{\delta\sqrt{n}} k^{-1} \sup_{k \geq 1} |n^{-1/2} \sum_{t=1}^{n} I(x'_t = k) - L(1, k/\sqrt{n})| = o_p(1). \tag{9}$$

Therefore,

$$A'_{n,\delta} = \log(n)^{-1} \sum_{k=1}^{\delta\sqrt{n}} k^{-1} L(1, k/\sqrt{n}) + o_p(1). \tag{10}$$

By the dominated convergence theorem we have that

$$A'_{n,\delta} \xrightarrow{d} L(1,0) \lim_{n\to\infty} \log(n)^{-1} \sum_{k=1}^{\delta\sqrt{n}} k^{-1} = (1/2)L(1,0) \overset{d}{=} (1/2)|Z|. \tag{11}$$

Therefore, the result for A_n follows. $\square$