Math 2162.02
Autumn 2017
Exam 3
11/8/2017
Time Limit: 55 Minutes

Name: ____________________
Name.##: ____________________
Lecturer: Jenny Sheldon
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Recitation Time: ____________

This exam contains 7 pages (including this cover page) and 5 questions. The total number of points is 100. Calculators are not permitted, and the exam is closed book and closed notes.

Do not open the exam until instructed to do so. Show all your work; solutions without appropriate supporting details may not be given credit. You need not simplify purely numerical expressions for your final answer unless otherwise instructed. Do not cheat.

Question	Points	Score
1	18	
2	20	
3	24	
4	18	
5	20	
Total:	100	

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1. (18 points) A table of values for a continuous function f with continuous second partial derivatives is shown. Use the table to answer the following questions.

(x,y)	$f_x(x,y)$	$f_y(x,y)$	$f_{xx}(x,y)$	$f_{yy}(x,y)$	$f_{xy}(x,y)$
$(0,0)$	-1	0	-23	2	8
$(-2,3)$	0	0	-15	1	4
$(2,2)$	0	3	-12	3	3
$(-2,-2)$	0	0	4	-2	-1
$(3,5)$	1	1	-8	-4	0

(a) (6 points) For $u = \langle -1, -1\rangle$, which is larger, $D_{\vec{u}}f(2,2)$ or $D_{\vec{u}}f(3,5)$?

Answer: Recall $D_{\vec{u}}f(x,y) = \nabla f(x,y) \cdot \frac{\vec{u}}{|\vec{u}|}$ and $\nabla f(x,y) = < f_x(x,y), f_y(x,y) >$

$D_{\vec{u}}f(2,2) = \nabla f(2,2) \cdot \frac{\vec{u}}{|\vec{u}|} = < f_x(2,2), f_y(2,2) > \cdot \frac{<-1,-1>}{\sqrt{2}} = <0,3> \cdot \frac{<-1,-1>}{\sqrt{2}} = -\frac{3}{\sqrt{2}}$

$D_{\vec{u}}f(3,5) = \nabla f(3,5) \frac{\vec{u}}{|\vec{u}|} = < f_x(3,5), f_y(3,5) > \cdot \frac{<-1,-1>}{\sqrt{2}} = <1,1> \cdot \frac{<-1,-1>}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$

$D_{\vec{u}}f(3,5)$ is larger in value, i.e. $D_{\vec{u}}f(3,5) > D_{\vec{u}}f(2,2)$

(b) (6 points) In what direction $\vec{u}$ is $D_{\vec{u}}f(2,2)$ maximized? What direction $\vec{v}$ is the direction of no change at $(2,2)$?

Answer: When $\vec{u} = \nabla f(2,2)$, $D_{\vec{u}}f(2,2)$ is maximized, $\vec{u} = <0,3>$

when $\vec{v} \perp \nabla f(2,2)$, f has no change, $\vec{v} = <1,0>$ (Note: the answer is not unique, any vector $\vec{v} \perp \nabla f(2,2)$ works)

(c) (6 points) Find and classify any critical points of f which are listed in the table.

Answer: Critical points: $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

Two critical points $(-2,3)$ and $(-2,-2)$

To classify them, calculate $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$

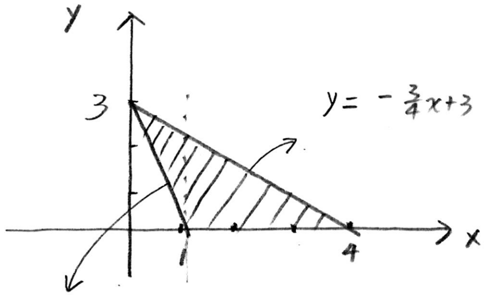
$D(-2,3) = (-15) \times 1 - 4^2 = -15 - 16 = -31 < 0$ $(-2,3)$ is a saddle

$D(-2,-2) = 4 \times (-2) - (-1)^2 = -8 - 1 = -9 < 0$ $(-2,2)$ is a saddle

2. (20 points) A thin triangular plate has vertices $(1,0)$, $(4,0)$, and $(0,3)$. Its density is given by the function $g(x,y) = y$.

(a) (4 points) Sketch and label the plate.

Answer:



$y = -\frac{3}{4}x + 3 \Leftrightarrow x = \frac{4}{3}(3-y)$

$y = -3x+3 \Leftrightarrow x = \frac{3-y}{3}$

(b) (8 points) Find the mass of the plate.

Answer:

$$\text{Mass} = \iint_R g(x,y)\,dA$$
$$= \int_0^3 \int_{\frac{3-y}{3}}^{\frac{4(3-y)}{3}} y \, dx\,dy$$
$$= \int_0^3 y \cdot \left(\frac{4(3-y)}{3} - \frac{3-y}{3}\right) dy$$
$$= \int_0^3 y(3-y)\,dy$$
$$= \int_0^3 3y - y^2 dy$$
$$= \frac{3y^2}{2} - \frac{y^3}{3} \Big|_0^3$$
$$= \frac{27}{2} - 9$$
$$= \frac{9}{2}$$

Method 2:

$$\iint_R g\,dA$$
$$= \int_0^1 \int_{-3x+3}^{-\frac{3}{4}x+3} g\,dy\,dx + \int_1^4 \int_0^{-\frac{3}{4}x+3} g\,dy\,dx$$

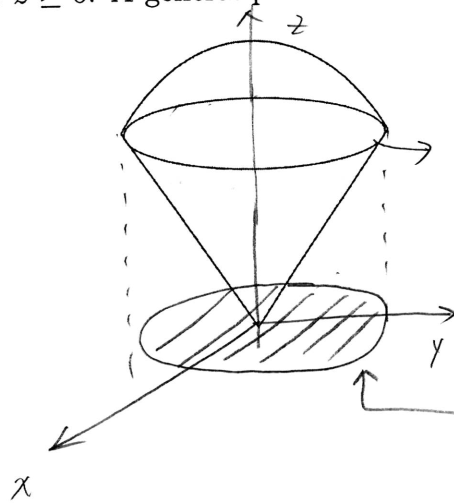
(c) (8 points) Set up, but do not evaluate, integral(s) which would compute the center of mass of the plate.

Answer: Center $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\iint_R x\,dA}{\text{Mass}} = \frac{2}{9} \cdot \int_0^3 \int_{\frac{3-y}{3}}^{\frac{4(3-y)}{3}} xy\,dx\,dy$$

$$\bar{y} = \frac{\iint_R y\,dA}{\text{Mass}} = \frac{2}{9} \int_0^3 \int_{\frac{3-y}{3}}^{\frac{4(3-y)}{3}} y^2 dx\,dy$$

3. (24 points) Let R be the three-dimensional region between the surfaces $z^2 = \frac{x^2}{3} + \frac{y^2}{3}$ and $x^2 + y^2 + z^2 = 12$ for $z \geq 0$. A general picture of this region is given below.



intersection:

$\frac{x^2}{3} + \frac{y^2}{3} = 12 - x^2 - y^2$

$\frac{4}{3}(x^2+y^2) = 12$

$x^2+y^2 = 9$

(a) (8 points) Set up, but do not evaluate, a triple integral in rectangular coordinates, for the volume of this region.

Answer: $V = \iiint\limits_D 1 \, dV$

$= \iint\limits_{R_{xy}} \int_{G(x,y)}^{H(x,y)} 1 \, dz \, dA$

$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{\frac{x^2+y^2}{3}}}^{\sqrt{12-x^2-y^2}} 1 \, dz \, dy \, dx$

(b) (8 points) Set up, but do not evaluate, a triple integral in cylindrical coordinates for the volume of this region.

Answer:

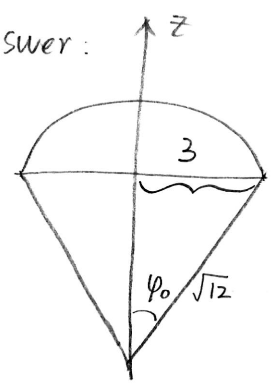
$$V = \iiint_D 1 \, dv$$

$$= \iint_{R_{polar}} \int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} 1 \, dz \, r dr d\theta$$

$$= \int_0^{2\pi} \int_0^3 \int_{\sqrt{\frac{r^2}{3}}}^{\sqrt{12-r^2}} 1 \, dz \, r dr d\theta.$$

(c) (8 points) Set up, but do not evaluate, a triple integral in spherical coordinates for the volume of this region.

Answer:



$$V = \iiint_D 1 \, dv$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^{\sqrt{12}} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$\sin\varphi_0 = \frac{3}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\varphi_0 = \frac{\pi}{3}$$

Alternatively, solve for φ_0.

$$z^2 = \frac{x^2}{3} + \frac{y^2}{3} \quad (\text{Cone})$$

$$(\rho\cos\varphi)^2 = \frac{1}{3}[(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2]$$

$$\cos^2\varphi = \frac{1}{3}\sin^2\varphi$$

$$\tan\varphi = \sqrt{3} \Rightarrow \varphi = \frac{\pi}{3}$$

4. (18 points) (a) (8 points) Find the equation of the plane tangent to the surface

$$x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$$

at the point $(1, 2, 3)$.

Answer: $F(x,y,z) = x^2 + \frac{y^2}{4} - \frac{z^2}{9} - 1$

$\nabla F(x,y,z) = \langle 2x, \frac{y}{2}, -\frac{2z}{9} \rangle$

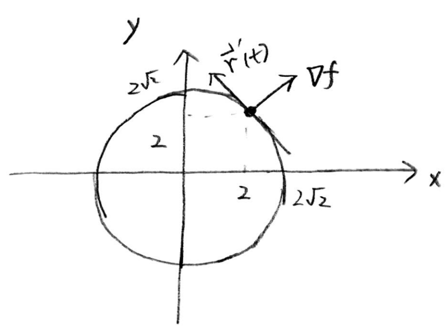
$\vec{n} = \nabla F(1,2,3) = \langle 2, 1, -\frac{2}{3} \rangle$

Tangent plane is

$2(x-1) + 1(y-2) - \frac{2}{3}(z-3) = 0$

(b) (10 points) Let $f(x, y) = x^2 + y^2$. Verify that ∇f is perpendicular to the level curve $f(x, y) = 8$ at the point $(2, 2)$. You might find it helpful to draw a picture!

Answer: Level curve: $x^2 + y^2 = 8$



$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$

$\nabla f(2,2) = \langle 4, 4 \rangle$

Method I:

slope of tangent line of $x^2+y^2=8$ at (x,y)

$k = \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x}{y}$

At $(2,2)$, slope $= -1$

The tangent vector is $\langle 1, -1 \rangle$ or $\langle -1, 1 \rangle$

Then $\langle 1, -1 \rangle \cdot \langle 4, 4 \rangle = 0$.

Method II: $x^2+y^2=8 \iff$

$\vec{r}(t) = \langle 2\sqrt{2}\cos t, 2\sqrt{2}\sin t \rangle$

$\vec{r}'(t) = \langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t \rangle$

$(2,2) \iff t = \frac{\pi}{4}$

$\vec{r}'(\frac{\pi}{4}) = \langle -2\sqrt{2}\frac{\sqrt{2}}{2}, 2\sqrt{2}\frac{\sqrt{2}}{2} \rangle = \langle -2, 2 \rangle$

And $\vec{r}'(\frac{\pi}{4}) \cdot \nabla f(2,2) = 0$

So ∇f is perpendicular to level curve.

5. (20 points) Consider the function

$$f(x,y,z) = 3x - y$$

on the region

$$R = \{(x,y,z) : x^2 + y^2 + z^2 \leq 2\}.$$

(a) (4 points) Explain why this function has absolute maximum and minimum values on the region.

Answer: Since f is continuous on the closed and bounded region R, f has absolute max and min on R. +2 +2

(b) (16 points) Find the absolute maximum and minimum values of f on the region R.

Answer:

step 1: Find critical points

$$\begin{cases} f_x = 0 \\ f_y = 0 \\ f_z = 0 \end{cases} \Leftrightarrow \begin{cases} 3 = 0 \\ -1 = 0 \\ 0 = 0 \end{cases} \Rightarrow \text{No critical point!}$$

step 2: Evaluate f on the boundary $x^2+y^2+z^2 = 2$

Substitution or parameterization won't work. Use Lagrange!

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0. \end{cases} \Leftrightarrow \begin{cases} 3 = \lambda \cdot 2x & (1) \\ -1 = \lambda \cdot 2y & (2) \\ 0 = \lambda \cdot 2z & (3) \\ x^2+y^2+z^2 = 2 & (4) \end{cases}$$

(1) $\Rightarrow$ $x = \frac{3}{2\lambda}$

(2) $\Rightarrow$ $y = \frac{-1}{2\lambda}$

(3) $\Rightarrow$ $z = 0$.

Use (4) : $(\frac{3}{2\lambda})^2 + (-\frac{1}{2\lambda})^2 + 0 = 2 \Leftrightarrow \lambda = \pm\frac{\sqrt{5}}{2}$

When $\lambda = \frac{\sqrt{5}}{2}$, one solution: $(\frac{3}{5}\sqrt{5}, -\frac{\sqrt{5}}{5}, 0)$

when $\lambda = -\frac{\sqrt{5}}{2}$, one solution: $(-\frac{3}{5}\sqrt{5}, \frac{\sqrt{5}}{5}, 0)$

$f_{max} = f(\frac{3}{5}\sqrt{5}, -\frac{\sqrt{5}}{5}, 0) = 2\sqrt{5}$

$f_{min} = f(-\frac{3}{5}\sqrt{5}, \frac{\sqrt{5}}{5}, 0) = -2\sqrt{5}$