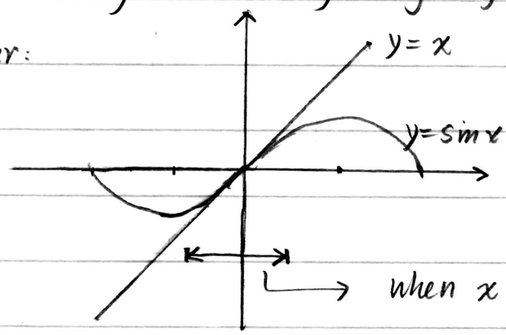
Math 2162.02 Written Homework 1

• Consider the approximation $\sin(x) \approx x$ for values of x near zero from several perspectives

1. Explain why $\sin x \approx x$ by using a graph.

Answer:



when x is near 0, $\sin x$ and x are close each other.

2. Explain why $\sin x \approx x$ by using a limit.

Answer: $\lim_{x\to 0}\left(\frac{\sin x}{x} - 1\right) = \left(\lim_{x\to 0}\frac{\sin x}{x}\right) - 1 = 1 - 1 = 0$ by L'Hospital's Rule

By the definition of limit, when x is close to 0, $\left|\frac{\sin x}{x} - 1\right| < \varepsilon$, where ε is a small number

So $1-\varepsilon < \left|\frac{\sin x}{x}\right| < 1+\varepsilon$ i.e. $(1-\varepsilon)|x| < |\sin x| < (1+\varepsilon)|x|$. Hence $\sin x \approx x$

3. Explain why $\sin x \approx x$ using a Taylor series.

Answer: The Taylor series of $\sin x$ with center 0 is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

When x is near 0, the higher order terms x^{2k+1} $(k \geq 1)$ are very small.

So $\sin x \approx x$.

4. How accurate is the approximation $\sin x \approx x$ on $[-0.1, 0.1]$

Answer: If we use the first term of Taylor series of $\sin x$ to approximate $\sin x$, the error can be estimated using remainder $R_1(x)$.

$$R_1(x) = \frac{f^{(2)}(c)}{2!}(x-0)^2$$ and

$$|R_1(x)| \leq \frac{|\sin(c)|}{2} x^2 \leq \frac{1}{2}\cdot(0.1)^2 = \frac{0.01}{2} = 0.005 \text{ for } x \in [-0.1, 0.1]$$

5. What interval would you choose to get an error no greater than 0.001?

Answer: Assume $|R_1(x)| \leq \frac{|\sin c|}{2} x^2 \leq \frac{1}{2}x^2 \leq 0.001$

$$x^2 \leq 0.002$$

$$-\sqrt{0.002} \leq x \leq \sqrt{0.002}$$

The interval is $[-\sqrt{0.002}, \sqrt{0.002}]$ #

Math 2162: Written Homework 1

Due: Thursday, September 14 in recitation

The goal of this assignment is two-fold: first, to give you an opportunity to think deeply about a subject we've been discussing, and second, to help you practice writing and communicating about mathematics. Towards these ends, the solutions you turn in for these problems should be easily understood by someone who is familiar with mathematics, but has not seen the assignment. It should be neat and professional - something you are proud to turn in. In particular, you should expect to write a few sentences explaining your conclusions.

You may work in groups, but each student needs to turn in their own assignment. If you do choose to work in a group, please write the name of everyone in your group.

We often discuss and use the approximation $\sin(x) \approx x$ for values of x near zero. We would like to consider this approximation from several perspectives.

1. Explain why $\sin(x) \approx x$ by using a graph.

2. Explain why $\sin(x) \approx x$ by using a limit.

3. Explain why $\sin(x) \approx x$ using a Taylor Series.

4. How accurate is the approximation $\sin(x) \approx x$ on the interval $[-0.1, 0.1]$?

5. What interval would you choose if you wanted to use the approximation $\sin(x) \approx x$ near zero, and required an error no greater than 0.001?