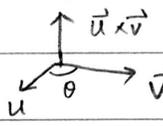


Sep. 26

§ 12.3-12.4

	$\vec{u} \cdot \vec{v}$	$\vec{u} \times \vec{v}$
①	$\vec{u} \cdot \vec{v}$ is a scalar $\vec{u} \cdot \vec{v} =  \vec{u}   \vec{v}  \cos \theta$	$\vec{u} \times \vec{v}$ is a vector ① Length: $ \vec{u} \times \vec{v}  =  \vec{u}   \vec{v}  \sin \theta$ ② Direction: right hand rule  $\vec{u} \times \vec{v}$ is orthogonal to the plane generated by $\vec{u}$ and $\vec{v}$
②	$\vec{u} = \langle x_1, y_1, z_1 \rangle$ $\vec{v} = \langle x_2, y_2, z_2 \rangle$ $\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2 + z_1 z_2$	$\vec{u} = \langle x_1, y_1, z_1 \rangle$ $\vec{v} = \langle x_2, y_2, z_2 \rangle$ $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$ $=  y_1 z_2 - y_2 z_1  \hat{i} -  x_1 z_2 - x_2 z_1  \hat{j} +  x_1 y_2 - x_2 y_1  \hat{k}$ $= (y_1 z_2 - y_2 z_1) \hat{i} - (x_1 z_2 - x_2 z_1) \hat{j} + (x_1 y_2 - x_2 y_1) \hat{k}$
③	$\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$ $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$	$\vec{u} \times \vec{v} = \vec{0} \Leftrightarrow \vec{u} \parallel \vec{v}$ $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
④	Work done by force $W = \vec{F} \cdot \vec{r}$	a. Area of parallelogram $ \vec{u} \times \vec{v} $ b. Torque $\vec{\tau} = \vec{r} \times \vec{F}$ c. Magnetic force $\vec{F} = q(\vec{v} \times \vec{B})$

Other important definitions

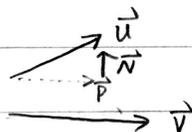
① Unit vector  $\parallel \vec{u}$  are  $\pm \frac{\vec{u}}{|\vec{u}|}$

★ ② Projection of  $\vec{u}$  on  $\vec{v}$ :  $\text{Proj}_{\vec{v}} \vec{u} = \underbrace{|\vec{u}| \cos \theta}_{\text{Scale } \vec{v}} \cdot \frac{\vec{v}}{|\vec{v}|} = \underbrace{\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}}_{\text{Scale } \vec{v}} \cdot \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \cdot \vec{v}$

Examples:

1. Let  $\vec{u} = 2\hat{i} - \hat{j}$   $\vec{v} = \hat{i} + \hat{j}$ . Find  $\vec{p}$  and  $\vec{n}$  such that  $\vec{u} = \vec{p} + \vec{n}$  and  $\vec{p} \parallel \vec{v}$   $\vec{n} \perp \vec{v}$

Answer:



✓ Answer:  $\vec{p} = \text{Proj}_{\vec{v}} \vec{u}$  and  $\vec{n} = \vec{u} - \vec{p}$

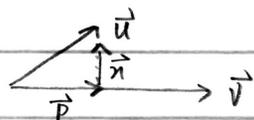
$$\begin{aligned} \vec{p} &= \text{Proj}_{\vec{v}} \cdot \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \cdot \vec{v} \\ &= \frac{1}{2} \cdot \langle 1, 1 \rangle \\ &= \langle \frac{1}{2}, \frac{1}{2} \rangle \end{aligned}$$

$$\begin{aligned} \vec{u} &= \langle 2, -1 \rangle \quad \vec{v} = \langle 1, 1 \rangle \\ \vec{u} \cdot \vec{v} &= 2 - 1 = 1 \\ \vec{v} \cdot \vec{v} &= 1 + 1 = 2 \end{aligned}$$

$$\vec{n} = \vec{u} - \vec{p} = \langle 2, -1 \rangle - \langle \frac{1}{2}, \frac{1}{2} \rangle = \langle \frac{3}{2}, -\frac{3}{2} \rangle \quad \#$$

★ HW5 #28.  $\vec{u} = \langle 4, -1, 1 \rangle$   $\vec{v} = \langle 1, 1, 3 \rangle$ , express  $\vec{u}$  as  $\vec{u} = \vec{p} + \vec{n}$ , where  $\vec{p} \parallel \vec{v}$  and  $\vec{n} \perp \vec{v}$

Answer:



$$\begin{aligned} \vec{p} &= \text{Proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \cdot \vec{v} \\ &= \frac{6}{11} \cdot \langle 1, 1, 3 \rangle = \langle \frac{6}{11}, \frac{6}{11}, \frac{18}{11} \rangle \end{aligned}$$

$$\vec{u} \cdot \vec{v} = 4 - 1 + 3 = 6$$

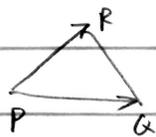
$$\vec{v} \cdot \vec{v} = 1 + 1 + 9 = 11$$

$$\vec{n} = \vec{u} - \vec{p} = \langle 4, -1, 1 \rangle - \langle \frac{6}{11}, \frac{6}{11}, \frac{18}{11} \rangle = \langle \frac{38}{11}, -\frac{17}{11}, -\frac{7}{11} \rangle \quad \#$$

Old midterm problem

Let  $P = (1, 1, 1)$   $Q = (0, 3, 1)$   $R = (0, 1, 4)$  Q: Find area of  $\triangle PQR$

Answer:



$$\text{Area of } \triangle PQR = \frac{1}{2} | \vec{PQ} \times \vec{PR} |$$

$$\vec{PQ} = \langle 0-1, 3-1, 1-1 \rangle = \langle -1, 2, 0 \rangle$$

$$\vec{PR} = \langle 0-1, 1-1, 4-1 \rangle = \langle -1, 0, 3 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

$$\frac{1}{2} | \vec{PQ} \times \vec{PR} | = \frac{1}{2} \sqrt{36 + 9 + 4} = \frac{1}{2} \sqrt{49} = \frac{7}{2} \quad \#$$

## § 12.5 Line and curves in space

① A Line through  $(x_0, y_0, z_0)$  with direction  $\vec{v} = \langle a, b, c \rangle$  is

Form I.  $\vec{r}(t) = (x_0, y_0, z_0) + \langle a, b, c \rangle t = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

Form II. 
$$\begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases}$$

Example:

Find a line through  $(4, -7, 0)$  and parallel to  $\vec{r} = \langle 1-t, 5t, 3+t \rangle$

Answer:  $\vec{v} = \langle 1, 5, 1 \rangle$

$$\vec{r}(t) = (4, -7, 0) + \langle 1, 5, 1 \rangle t$$

$$= \langle 4-t, -7+5t, t \rangle \quad \#$$

More practice:

① Find an equation of line passing through  $(1, 2, -1)$  and  $(3, 4, 0)$

Answer:  $\vec{v} = \langle 3-1, 4-2, 0-(-1) \rangle = \langle 2, 2, 1 \rangle$

$$\vec{r}(t) = (1, 2, -1) + \langle 2, 2, 1 \rangle t$$

$$= \langle 1+2t, 2+2t, -1+t \rangle \quad \#$$

② Find an equation of line that  $\perp$  both  $\vec{u} = \vec{i} + \vec{k}$  and  $\vec{v} = \vec{j} - \vec{k}$  that passes through  $(-1, 2, 3)$

Answer:  $\vec{v}_1 = \langle 1, 0, 1 \rangle$   $\vec{v}_2 = \langle 0, 1, -1 \rangle$

$$\vec{v} = \vec{v}_1 \times \vec{v}_2 = \langle -1, 1, 1 \rangle$$

$$\vec{r}(t) = (-1, 2, 3) + \vec{v}t = (-1, 2, 3) + \langle -1, 1, 1 \rangle t$$

$$= \langle -1-t, 2+t, 3+t \rangle \quad \#$$

## § 12.6 Vector valued function

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

Geometric meaning:  $\vec{r}(t)$  can be viewed as a trajectory of a particle in 3D space and  $t$  is time.

### Related Problems

① Domain of  $\vec{r}(t)$  is the intersection of the domains of  $f(t), g(t), h(t)$

② Limit  $\lim_{t \rightarrow t_0} \vec{r}(t) = \langle \lim_{t \rightarrow t_0} f(t), \lim_{t \rightarrow t_0} g(t), \lim_{t \rightarrow t_0} h(t) \rangle$

③ Derivative  $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

④ Integral:  $\int \vec{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle$

Derivative Rules: Thm 12.7 in § 12.6

An interesting problem: Intersection and collision of two curves  $\vec{r}(t)$  and  $\vec{r}(s)$

Note: You can think  $\vec{r}(t)$  and  $\vec{r}(s)$  are trajectories of two particles. The intersections

are common pts on the trajectories, while the collisions are not only common pts, the particles also have to be at the common pts at the same time.

Example:  $\vec{r}(t) = \langle 3t+4, 5t+1, 3t+6 \rangle$

$$\vec{r}(s) = \langle s+3, 3s-2, -5s+1 \rangle$$

① Intersections

$$\begin{cases} 3t+4 = s+3 \\ 5t+1 = 3s-2 \\ 3t+6 = -5s+1 \end{cases} \Rightarrow \begin{cases} t=0 \\ s=1 \end{cases}$$

Intersection:  $(4, 1, 6)$

② Collision? No, because two particles do not reach the intersection at the same time. #