

Sep. 19 S11.1-11.3

Topic 1: Parametric equations

In general, $x = x(t)$ and $y = y(t)$ $a \leq t \leq b$

Determine the direction: step 1: Fix a starting point, usually $t=0$.

step 2: check the mono

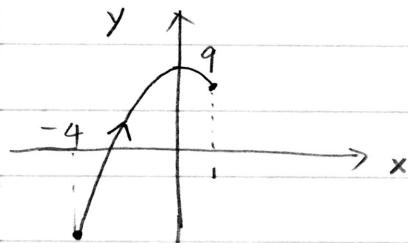
$$\text{Derivatives: } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Examples:

HW4 #8. Cartesian \rightarrow parametric

$$y = -3x^2 + 9 \quad -4 \leq x \leq 1$$

$$\begin{aligned} \text{Answer: } x &= t \\ &\quad -4 \leq t \leq 1 \\ y &= -3t^2 + 9 \end{aligned}$$



Direction: ?

As $t \uparrow$, $x \uparrow$

$y = -3x^2 + 9 \uparrow$ since $|x| \downarrow$.

Remark: Summary:

#1 ① Line: $A(x_0, y_0) \rightarrow B(x_1, y_1)$ $\vec{r} = (x_1 - x_0, y_1 - y_0)$

$$x(t) = x_0 + (x_1 - x_0)t$$

$$y(t) = y_0 + (y_1 - y_0)t \quad 0 \leq t \leq 1 \quad \text{Direction } A \rightarrow B$$

#2 ② Circle $(x - x_0)^2 + (y - y_0)^2 = r^2$

$$x(t) = x_0 + r \cos t$$

$$y(t) = y_0 + r \sin t \quad 0 \leq t \leq 2\pi \quad \text{Direction: counterclockwise}$$

#3 ③ Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x(t) = a \cos t$$

$$0 \leq t \leq 2\pi \quad \text{counterclockwise}$$

$$y(t) = b \sin t$$

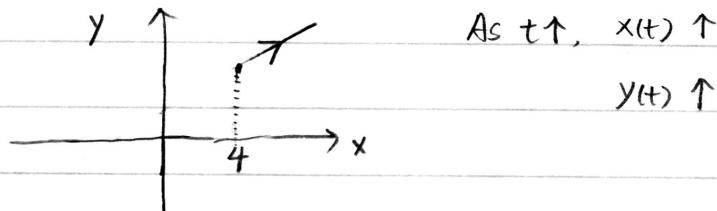
HW4 # 3 Parametric \rightarrow Cartesian

$$x(t) = 4e^t \quad y(t) = 1 + e^t \quad t \geq 0$$

Answer: eliminate t

$$e^t = \frac{x}{4} \text{ replace } e^t \text{ by } \frac{x}{4} \text{ in } y = t + e^t$$

$$y = t + \frac{x}{4} \quad x = 4e^t = 4$$



Topic 2 Polar coordinates

- In general

Cartesian	$r = \sqrt{x^2 + y^2}$	Polar
(x, y)	$\theta = \arctan \frac{y}{x}$	(r, θ)
	$x = r \cos \theta$	
	$y = r \sin \theta$	

Example: $(x, y) = (\sqrt{3}, 1)$ Find (r, θ)

$$\text{Answer: } r = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2$$

$$\tan \theta = \frac{y}{x} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \theta = \frac{\pi}{6}$$

- Conversion: $(r, \theta) = (-r, \theta + \pi)$ $(r, \theta) = (r, \theta + 2\pi)$

Example: $(-3, -\frac{3\pi}{4})$ equivalent expression.

$$(-3, -\frac{3\pi}{4}) = (3, -\frac{3\pi}{4} + \pi) = (3, \frac{\pi}{4})$$

Examples HW4 #16 #21

- #16. $r = -10 \cos \theta - 10 \sin \theta$. Find Cartesian equation.

$$\text{Answer: } x = r \cos \theta, y = r \sin \theta$$

$$r = -10 - \frac{x}{r} - 10 - \frac{y}{r}$$

$$r^2 = -10x - 10y$$

$$x^2 + y^2 = -10x - 10y$$

$$x^2 + 10x + y^2 + 10y = 0$$

$$(x+5)^2 + (y+5)^2 = 50$$

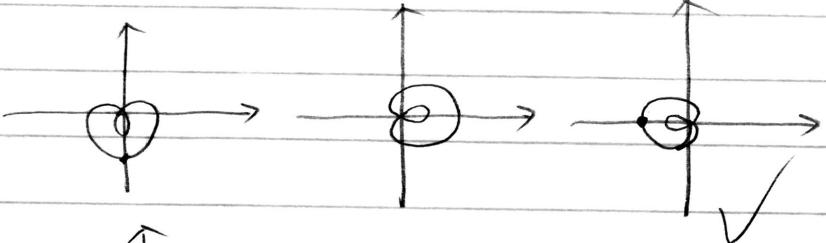
- # 21 $x = 9y^2$. Find polar form

$$\text{Answer: } x \cos \theta = 9(r \sin \theta)^2 = 9r^2 \sin^2 \theta \quad r = \frac{1}{9} \cos \theta \frac{1}{\sin^2 \theta} = \frac{1}{9} \cot \theta \csc \theta.$$

- Plotting:

Example. HW4 #18

$$r = 4 - 8 \cos \theta$$

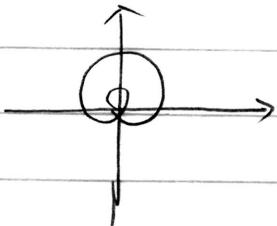


Answer:

check special points

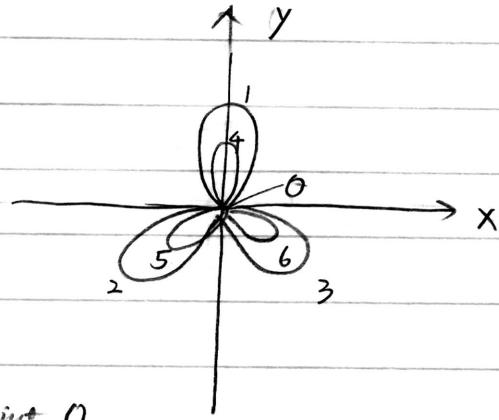
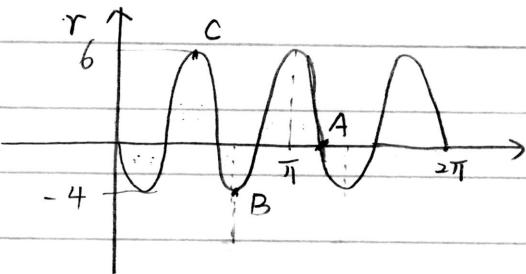
$$\theta = 0 \quad r = 4 - 8 = -4$$

$$(-4, 0) = (4, \pi)$$



- Cartesian to polar

$$r = 1 - 5 \sin(3\theta)$$



Answer: $A = (0, \frac{4\pi}{3})$ $r=0 \Rightarrow$ point 0

$B = (-4, \frac{5\pi}{6}) = (4, \frac{5\pi}{6} + \pi) \Rightarrow$ point 6

$C = (6, \frac{1}{2}\pi) \Rightarrow$ point 1

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Topic 3 Calculus in Polar

Given $r = f(\theta)$

$$\textcircled{1} \text{ Derivative } \frac{dy}{dx} = \frac{\frac{dy/d\theta}{dx/d\theta}}{\frac{(f(\theta) \cdot \sin \theta)'}{(f(\theta) \cos \theta)'}} = \frac{\sin \theta f'(\theta) + f(\theta) \cos \theta}{\cos \theta f'(\theta) - \sin \theta f(\theta)}$$

Related problems: tangent line

$$\textcircled{2} \text{ Area bdd by polar curves. } \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta)^2 - g(\theta)^2) d\theta$$

- key points: α, β : intersections

$f(\theta)$ outer curve $g(\theta)$ inner curves

use symmetry.

HW 5

Examples:

#3. $r = 5 + 5\sin\theta = f(\theta)$

① Horizontal tangent line

Answer: $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{\sin\theta f'(\theta) + \cos\theta f(\theta)}{\cos\theta f'(\theta) - \sin\theta f(\theta)} = \frac{\sin\theta(5\cos\theta) + \cos\theta(5+5\sin\theta)}{\cos\theta(5\cos\theta) - \sin\theta(5+5\sin\theta)}$$

$$\frac{dy}{dx} = 0 \Leftrightarrow \text{numerator} = 0 \text{ and denominator} \neq 0$$

$$5\sin\theta\cos\theta + 5\cos\theta + 5\sin\theta\cos\theta = 0$$

$$10\sin\theta\cos\theta + 5\cos\theta = 0$$

$$5\cos\theta(2\sin\theta + 1) = 0$$

$$\cos\theta = 0 \text{ or } \sin\theta = -\frac{1}{2}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\theta = \frac{\pi}{2} \quad r = 5 + 5\sin\frac{\pi}{2} = 10$$

$$\theta = \frac{7\pi}{6} \quad r = 5 + 5\sin\frac{7\pi}{6} = 0$$

$$\theta = \frac{11\pi}{6} \quad r = \frac{5}{2} \quad \theta = \frac{11\pi}{6} \quad r = \frac{5}{2}$$

But $\cos\theta(5\cos\theta) - \sin\theta(5+5\sin\theta) \neq 0 \Leftrightarrow$

$$5\cos^2\theta - 5\sin\theta - 5\sin^2\theta = 0$$

$$5 - 5\sin^2\theta - 5\sin\theta - 5\sin^2\theta = 0$$

$$10\sin^2\theta + 5\sin\theta - 5 = 0$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(\sin\theta + 1)(2\sin\theta - 1) = 0$$

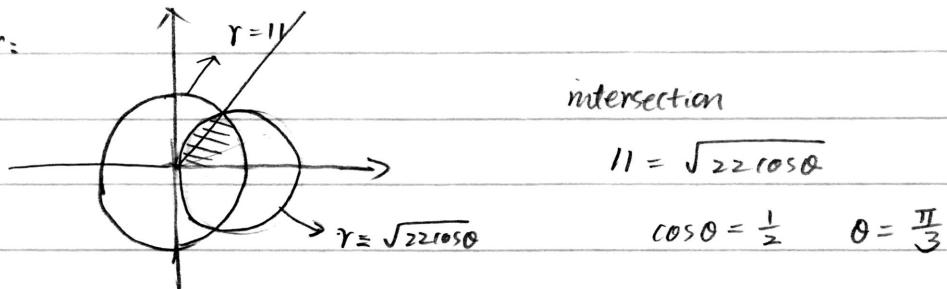
$$\sin\theta \neq -1 \quad \sin\theta \neq \frac{1}{2} \Rightarrow \text{cannot include } \theta = \frac{\pi}{2}$$

② Vertical tangent line? (Practice)

5. Find area inside both curves.

$$r = \sqrt{2\cos\theta} \quad r = \sqrt{11}$$

Answer:



$$\begin{aligned} \text{Area} &= \int_{0}^{\frac{\pi}{3}} \frac{(\sqrt{11})^2}{2} d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(\sqrt{22\cos\theta})^2}{2} d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{11}{2} d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 11\cos\theta d\theta \\ &= \frac{11}{2} \cdot \frac{\pi}{3} + 11\sin\theta \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \frac{11\pi}{6} + 11 \cdot \left(1 - \frac{\sqrt{3}}{2}\right) \quad \# \end{aligned}$$

Recall: Double angle formulae

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$