

Quiz 7

MATH 2162.02

RECITATION TIME:

NAME:

Problem 1.(10 points)

Let

$$\phi(x, y) = \frac{x^2}{2} + y^2.$$

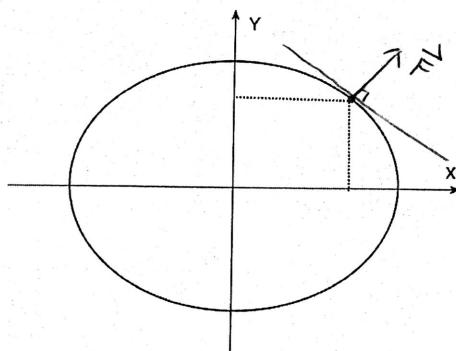
(a) Find the gradient field $\mathbf{F} = \nabla\phi$ for the potential function ϕ .

Answer: $\vec{\mathbf{F}} = \nabla\phi = \langle \phi_x, \phi_y \rangle = \langle x, 2y \rangle$

(b) Find a parametric description for the curve $C: \frac{x^2}{2} + y^2 = 1$ oriented counterclockwise in the form of $\vec{r}(t) = \langle x(t), y(t) \rangle$.

Answer: $\vec{r}(t) = \langle \sqrt{2} \cos t, \sin t \rangle \quad 0 \leq t \leq 2\pi$

(c) Sketch the gradient vector at $(1, \frac{\sqrt{2}}{2})$ on the equipotential curve $C: \frac{x^2}{2} + y^2 = 1$ and verify that the gradient vector is orthogonal to C .



$$\vec{\mathbf{F}}(1, \frac{\sqrt{2}}{2}) = \langle 1, \sqrt{2} \rangle$$

$$\vec{r}(t) = \langle \sqrt{2} \cos t, \sin t \rangle = \langle 1, \frac{\sqrt{2}}{2} \rangle$$

$$\text{solve for } t \quad \sqrt{2} \cos t = 1 \quad \cos t = \frac{\sqrt{2}}{2} \quad t = \frac{\pi}{4}$$

$$\vec{r}'(t) = \langle -\sqrt{2} \sin t, \cos t \rangle$$

$$\vec{r}'(\frac{\pi}{4}) = \langle -\sqrt{2} \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \langle -1, \frac{\sqrt{2}}{2} \rangle$$

$$\text{At } (1, \frac{\sqrt{2}}{2}), \quad \vec{\mathbf{F}} \cdot \vec{r}' = -1 + \sqrt{2} \cdot \frac{\sqrt{2}}{2} = -1 + 1 = 0$$

(d) For the same curve C , convert the following line integral to an ordinary integral with respect to parameter t and evaluate it.

Answer: $\int_C xy \, ds$

$$\int_C xy \, ds = \int_0^{2\pi} x(t)y(t) |\vec{r}'(t)| \, dt$$

$$= \int_0^{2\pi} \sqrt{2} \cos t \sin t \sqrt{2 \sin^2 t + \cos^2 t} \, dt$$

$$= \int_0^{2\pi} \sqrt{2} \cos t \sin t \sqrt{1 + \sin^2 t} \, dt$$

Let $u = 1 + \sin^2 t$

$$du = 2 \sin t \cos t \, dt$$

$$\int \sqrt{2} \cos t \sin t \sqrt{1 + \sin^2 t} \, dt$$

$$= \int \frac{\sqrt{2}}{2} \sqrt{u} \, du$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{2}{3} u^{\frac{3}{2}}$$

$$= \frac{\sqrt{2}}{3} (1 + \sin^2 t)^{\frac{3}{2}}$$

$$\text{Answer} = \frac{\sqrt{2}}{3} (1 + \sin^2 t)^{\frac{3}{2}} \Big|_0^{2\pi}$$

$$= 0$$