

Quiz 5

MATH 2162.02

RECITATION TIME:

NAME:

Problem 1. (10 points.) Consider the function

$$f(x, y) = \frac{1}{3}x^3 + x^2 + y^2$$

and the region

$$R = \{(x, y) : x^2 + y^2 \leq 1\}.$$

Why does $f(x, y)$ have both an absolute minimum and an absolute maximum on the region R ? Find the critical points of $f(x, y)$ on R and identify the points where $f(x, y)$ achieves its minimum and maximum values.

Answer:

Step 1 Find critical points

$$\begin{cases} f_x = x^2 + 2x = 0 \\ f_y = 2y = 0 \end{cases}$$

$$x = 0 \text{ or } -2$$

$$y = 0$$

$$(0, 0), (-2, 0)$$

But $(-2, 0)$ is not on Region R .

$$f(0, 0) = 0$$

Step 2 Evaluate f on the boundary of R

Method I.

On the boundary $x^2 + y^2 = 1$

$$\begin{aligned} f(x, y) &= \frac{1}{3}x^3 + x^2 + y^2 \\ &= \frac{1}{3}x^3 + 1 \end{aligned}$$

$$\text{And } -1 \leq x \leq 1$$

$$f_{\max} = \frac{1}{3} + 1 = \frac{4}{3} \text{ when } x = 1, y = 0$$

$$f_{\min} = -\frac{1}{3} + 1 = \frac{2}{3} \text{ when } x = -1, y = 0$$

Method II.

On the boundary $x^2 + y^2 = 1$

$$\text{Let } x = \cos t, y = \sin t \quad 0 \leq t \leq 2\pi$$

$$\begin{aligned} f(x, y) &= f(t) = \frac{1}{3}(\cos t)^3 + (\cos t)^2 + (\sin t)^2 \\ &= \frac{1}{3}(\cos t)^3 + 1 \end{aligned}$$

$$f_{\max} = \frac{1}{3} + 1 = \frac{4}{3} \text{ when } t = 0, 2\pi \Leftrightarrow x = 1, y = 0$$

$$f_{\min} = -\frac{1}{3} + 1 = \frac{2}{3} \text{ when } t = \pi \Leftrightarrow x = -1, y = 0$$

Method III. Lagrange multiplier method

Objective function $f(x, y)$

Constraint: $g(x, y) = x^2 + y^2 - 1 = 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} x^2 + 2x = 2x\lambda & (1) \\ 2y = 2y\lambda & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

$$(2) \Rightarrow y = 0 \text{ or } \lambda = 1$$

$$\text{Case I } y = 0$$

$$(3) \Rightarrow x = \pm 1$$

$$\text{Case II } \lambda = 1$$

$$(1) \Rightarrow x = 0$$

$$(3) \Rightarrow y = \pm 1$$

$$f_{\max} = f(1, 0) = \frac{4}{3}, f_{\min} = f(-1, 0) = \frac{2}{3}$$

Step 3: Overall $f_{\max} = \frac{4}{3}, f_{\min} = 0$