

§ 13.3

Def: $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)$ exists if the limit exists and equals along all paths $(x,y) \rightarrow (x_0,y_0)$

Standard Procedure to evaluate the limit.

Step 1: Plug in (x_0,y_0) . If nothing goes wrong, then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = f(x_0,y_0)$

If the denominator = 0 at (x_0,y_0) , then go to step 2

Step 2: Simplify $f(x,y)$ using factorization or rationalization

Or use some other techniques, e.g.: change of variable, polar coordinates

Then plug in (x_0,y_0) or new values (if you use change of variable)

If you still cannot plug in (x_0,y_0) - then go to step 3. Note: once you go to step 3, you have assumed that the limit DNE

Step 3: Use two-path test to show that the limit DNE

Two path test: choose two different paths and evaluate the limit along these paths to get two different values

Examples

HW7 # 19 $\lim_{(x,y) \rightarrow (4,e^2)} \ln \sqrt{xy} = \ln \sqrt{4 \cdot e^2} = \ln \sqrt{4} + \ln e^2 = \ln 2 + 2$

Midterm

$$\lim_{(x,y) \rightarrow (14,2)} \frac{x^2 - 7xy}{x - 7y} = \lim_{(x,y) \rightarrow (14,2)} \frac{x(x-7y)}{x-7y} = \lim_{(x,y) \rightarrow (14,2)} x = 14$$

Midterm $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - 7y}{x^2 + 7y}$ cannot be simplified

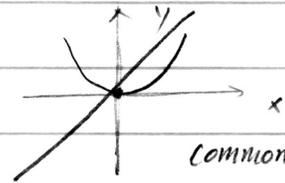
Let $x=0$ and along $x=0$

$$\lim_{y \rightarrow 0} \frac{-7y}{7y} = -1$$

Let $y=0$ and along $y=0$

$$\lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1$$

> different



commonly chosen paths

$$\begin{array}{ll} x=0 & y = \pm x \\ y=0 & y = x^2 \end{array}$$

HW7 # 21, 22, 25

The limit DNE by two-path test →

HW7 # 20 $\lim_{(x,y) \rightarrow (1,3)} \frac{\sqrt{x+y} - 2}{x+y - 4} = \lim_{(x,y) \rightarrow (1,3)} \frac{(\sqrt{x+y} - 2)(\sqrt{x+y} + 2)}{(x+y - 4)(\sqrt{x+y} + 2)} = \lim_{(x,y) \rightarrow (1,3)} \frac{x+y - 4}{(x+y - 4)(\sqrt{x+y} + 2)}$

$$= \lim_{(x,y) \rightarrow (1,3)} \frac{1}{\sqrt{x+y} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

HW7 # 24

$$\lim_{(x,y,z) \rightarrow (4,4,1)} \frac{x - \sqrt{xy} - \sqrt{xz} + \sqrt{yz}}{x - \sqrt{xy} + \sqrt{xz} - \sqrt{zy}}$$

$$= \lim_{(x,y,z) \rightarrow (4,4,1)} \frac{\sqrt{x}(\sqrt{x}-\sqrt{y}) - \sqrt{z}(\sqrt{x}-\sqrt{y})}{\sqrt{x}(\sqrt{x}-\sqrt{y}) + \sqrt{z}(\sqrt{x}-\sqrt{y})} = \lim_{(x,y,z) \rightarrow (4,4,1)} \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}-\sqrt{z})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{z})}$$

$$= \lim_{(x,y,z) \rightarrow (4,4,1)} \frac{\sqrt{x}-\sqrt{z}}{\sqrt{x}+\sqrt{z}} = \frac{\sqrt{4}-\sqrt{1}}{\sqrt{4}+\sqrt{1}} = \frac{1}{3}$$

HW7 # 26

$$\lim_{(x,y) \rightarrow (2,0)} \frac{1-\cos y}{2xy^2}$$

Recall $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x)g(y) = f(x_0)g(y_0)$

$$= \lim_{(x,y) \rightarrow (2,0)} \frac{1}{2x} \cdot \frac{1-\cos y}{y^2}$$

$$= \lim_{(x,y) \rightarrow (2,0)} \frac{1}{2x} \cdot \frac{2\sin^2(\frac{y}{2})}{4(\frac{y}{2})^2}$$

Note $\lim_{y \rightarrow 0} \frac{\sin^2(\frac{y}{2})}{(\frac{y}{2})^2} = \lim_{y \rightarrow 0} \left(\frac{\sin(\frac{y}{2})}{\frac{y}{2}}\right)^2 = 1^2 = 1$

$$= \frac{1}{4} \cdot \frac{2}{4} = \frac{1}{8}$$

HW7 # 27

$$\lim_{(x,y) \rightarrow (4,0)} \frac{(4xy)^{xy}}{-}$$

Let $u = xy$, $(x,y) \rightarrow (4,0) \Leftrightarrow u \rightarrow 0$

$$\lim_{u \rightarrow 0} (4u)^u = \lim_{u \rightarrow 0} e^{\ln(4u)^u} = \lim_{u \rightarrow 0} e^{u \ln(4u)}$$

$$= \lim_{u \rightarrow 0} e^{\frac{\ln(4u)}{\frac{1}{u}}} = \lim_{u \rightarrow 0} e^{\frac{4}{4u} \cdot \frac{1}{u}} = \lim_{u \rightarrow 0} e^{-u} = e^0 = 1$$

Def: Continuity.

$f(x,y)$ is continuous at (a,b) if ① f is defined at (a,b)

② $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ exists ③ $f(x_0,y_0) = \lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)$

Example: HW7 # 28

$$f(x,y) = \begin{cases} \frac{9 \sin(x^2+y^2-1)}{x^2+y^2-1} & \text{if } x^2+y^2 \neq 1 \\ b & \text{if } x^2+y^2 = 1 \end{cases} \quad \text{Find } b \text{ such that } f \text{ is cts in } \mathbb{R}^2$$

Answer: ① \checkmark defined

② $\lim_{x^2+y^2 \rightarrow 1} f(x,y)$ exists

③ the limit in ② = b

$$\lim_{x^2+y^2 \rightarrow 1} f(x,y) = \lim_{u \rightarrow 0} \frac{9 \sin u}{u} \quad \text{Let } u = x^2+y^2-1$$

$$= 9 = b$$

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$$\text{HW7 \# 22 } f(x,y) = \begin{cases} \frac{xy}{9x^2+2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Find the points where f is continuous

Answer: It's easy to see that f is cts except $(0,0)$

Now, check the continuity at $(0,0)$

① defined at $(0,0)$ ✓

$$\textcircled{2} \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{9x^2+2y^2}$$

$$\text{Let } x=0 \quad \lim_{y \rightarrow 0} \frac{0}{2y^2} = 0$$

$$\text{Let } x=y \quad \lim_{x \rightarrow 0} \frac{x^2}{9x^2+2x^2} = \lim_{x \rightarrow 0} \frac{x^2}{11x^2} = \frac{1}{11}$$

> different

The limit DNE.

So f is cts at all points except $(0,0)$

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