

§ 13.8 . 13.9

Topic: Classify critical points using 2nd derivative test.

Def: Critical points: the points that $f_x = f_y = 0$ or derivative DNE

2nd derivative test: Suppose (a,b) is a critical point. Define

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$

Local Extrema

- ① If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, f has a local min at (a,b)
- ② If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, f has a local max at (a,b)
- ③ If $D(a,b) < 0$, f has a saddle pt at (a,b)
- ④ If $D(a,b) = 0$, the test is inconclusive

Practice: HW8 #21 - 26

Topic 2: Extrema of objective function f subject to constraint $g=0$

Method: Lagrange multiplier.

Step 1: Set up $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$ and solve for (x,y) and λ

Step 2: Evaluate f at the points (x,y) found in step 1 to find max/min

Practice: HW9 #1 - 7

Topic 3 Global Extrema: Step 1: Find local extrema of f on open region (evaluate f at critical pts)

Step 2: Find extrema of f on the boundary (subject to boundary constraint)

Method: parameterize the boundary (circle, ellipse, parametric curve)

reduce one variable by substitution. (triangle, straight line)

Lagrange multiplier. (single equation as constraint) boundary

Step 3: Compare function value in step 1 and 2 and choose max/min

Examples HW8 #27

$$R: \{(x,y) : (x-1)^2 + y^2 \leq 1\} \quad f(x,y) = 3x^2 - 6x + 6y^2 + 6$$

(a) Find the critical pts on the interior of region R .

Answer: $\begin{cases} f_x = 6x - 6 = 0 \\ f_y = 12y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases} (1, 0)$

Does $(1, 0)$ lie on the interior of region R ? Yes!

And $f(1, 0) = 3 - 6 + 6 = 3$

⑥ Find the absolute max and min of f on R .

Answer: Method 1: Parameterize the boundary $(x-1)^2 + y^2 = 1$

$$\begin{cases} x = 1 + \cos \theta \\ y = \sin \theta \end{cases}$$

$$\begin{aligned} \text{On the boundary, } f(\theta) &= 3(1+\cos\theta)^2 - 6(1+\cos\theta) + 6(\sin\theta)^2 + 6 \\ &= 3 + 3\cos^2\theta + 6\cos\theta - 6 - 6\cos\theta + 6\sin^2\theta + 6 \\ &= 3 + (3\cos^2\theta + 3\sin^2\theta) + 3\sin^2\theta \\ &= 6 + 3\sin^2\theta \end{aligned}$$

$f(\theta)$ reaches max when $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

$f(\theta)$ reaches min when $\theta = 0, \pi$ and 2π .

At $\theta = \frac{\pi}{2}, x = 1, y = 1, f(1, 1) = 9$

At $\theta = \frac{3\pi}{2}, x = 1, y = -1, f(1, -1) = 9$

At $\theta = 0, x = 2, y = 0, f(2, 0) = 6$

At $\theta = \pi, x = 0, y = 0, f(0, 0) = 6$

At $\theta = 2\pi, x = 2, y = 0, f(2, 0) = 6$

Compared with $f(1, 0) = 3$, f has absolute max 9 at $(1, 1)$ and $(1, -1)$

f has absolute min at $(1, 0)$

Method 2: Lagrange multiplier

Objective function $f(x, y) = 3x^2 - 6x + 6y^2 + 6$

Constraint: $g(x, y) = (x-1)^2 + y^2 - 1 = 0$

$$\begin{array}{l} \text{Set up } \begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Leftrightarrow \begin{cases} 6x - 6 = \lambda \cdot 2(x-1) \\ 12y = \lambda \cdot 2y \\ (x-1)^2 + y^2 - 1 = 0 \end{cases} \end{array}$$

① $\Rightarrow x = 1$ or $\lambda = 3$

If $x = 1$, ③ $\Rightarrow y = \pm 1$ $(1, 1)$ and $(1, -1)$

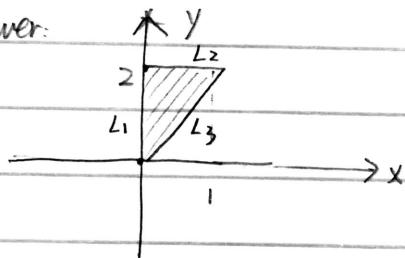
If $\lambda = 3$ ② $\Rightarrow y = 0$ and ③ $\Rightarrow x = 2$ or 0 $(2, 0)$ $(0, 0)$

Then evaluate f at these 4 points. You will get same answer as in method 1.

HW8 #28

Example 2: Find absolute max/min on the closed region bounded by the triangle with vertices $(0,0)$, $(0,2)$ and $(1,2)$ in the first quadrant.

Answer:



$$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$$

Step 1: Find critical pts on open region

$$\begin{cases} f_x = 4x - 4 = 0 \\ f_y = 2y - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$$

Note: $(1,2)$ is on the boundary, not the interior

$$f(1,2) = 2 - 4 + 4 - 8 + 1 = -5$$

Step 2: Evaluate f on the boundary.

$$\text{On } L_1: x=0, 0 \leq y \leq 2$$

$$f(y) = y^2 - 4y + 1 = (y-2)^2 - 3$$

$$f_{\max} = f(0,0) = 4 - 3 = 1 \quad \checkmark$$

$$f_{\min} = f(0,2) = -3$$

$$\text{On } L_2: 0 \leq x \leq 1 \text{ and } y = 2$$

$$f(x) = 2x^2 - 4x + 4 - 8 + 1 = 2x^2 - 4x - 3 = 2(x-1)^2 - 5$$

$$f_{\max} = f(0,2) = 2 - 5 = -3$$

$$f_{\min} = f(1,2) = -5$$

$$\text{On } L_3: y = 2x \text{ and } 0 \leq x \leq 1$$

$$f(x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1 = 6x^2 - 12x + 1 = 6(x-1)^2 - 5$$

$$f_{\max} = f(0,0) = 1 \quad \checkmark$$

$$f_{\min} = f(1,2) = -5$$

$$\text{Overall, } f_{\max} = f(0,0) = 1 \quad f_{\min} = f(1,2) = -5$$

Compared with result in step 1

$$f_{\min} = -5 \quad f_{\max} = 1$$

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Similar HW8 #29

Example 3: Find the point on the plane $2x+5y+z=8$ that is nearest to $(2, 0, 1)$

Answer: Objective: $f(x, y, z) = (x-2)^2 + y^2 + (z-1)^2$ (Squared distance)

Constraint: $g(x, y, z) = 2x+5y+z-8 = 0$

Set up $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} 2(x-2) = 2\lambda & (1) \\ 2y = 5\lambda & (2) \\ 2(z-1) = \lambda & (3) \\ 2x+5y+z-8=0 & (4) \end{cases}$$

$$(1) \Rightarrow x = \lambda + 2 \quad (2) \Rightarrow y = \frac{5\lambda}{2} \quad (3) \Rightarrow z = \frac{\lambda}{2} + 1$$

$$2(\lambda+2) + 5\left(\frac{5\lambda}{2}\right) + \left(\frac{\lambda}{2} + 1\right) - 8 = 0 \Rightarrow \lambda = \frac{1}{5}$$

$$\Rightarrow x = \lambda + 2 = \frac{1}{5} + 2 = 2\frac{1}{5}$$

$$y = \frac{5\lambda}{2} = \frac{1}{2}$$

$$z = \frac{\lambda}{2} + 1 = \frac{1}{10} + 1 = 1\frac{1}{10}$$

$$d_{\min} = \sqrt{(x-2)^2 + y^2 + (z-1)^2} = \sqrt{\frac{3}{10}}$$

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Similar practice: HW9 #3. #4