

§ 13.6 - § 13.7

Def 1: Gradient: Let f be differentiable at (a, b) . The gradient of f at (a, b) is

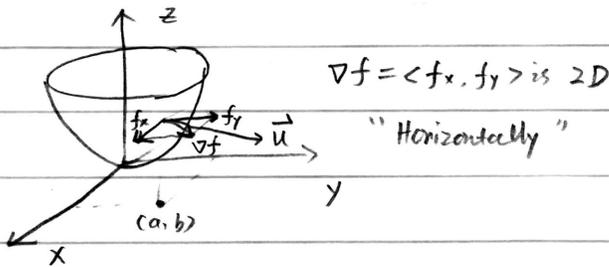
$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$$

Def 2: Directional derivative of f at (a, b) in the direction of \vec{u} is

$$D_{\vec{u}} f(a, b) = \nabla f(a, b) \cdot \frac{\vec{u}}{|\vec{u}|}$$

Demonstration:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{elliptic paraboloid}$$

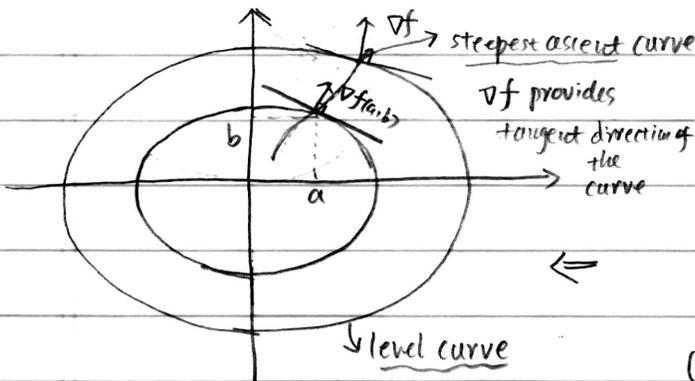


Directional derivative in \vec{u}

$$\frac{\nabla f(a, b) \cdot \vec{u}}{|\vec{u}|} = \text{Scal}_{\vec{u}} \nabla f(a, b)$$

Meaning: rate of change in \vec{u} direction.

projection on xy-plane
level curve through (a, b) point



Thm 13.11

(steepest ascent)

- ① f has max rate of increase at (a, b) in the direction of $\nabla f(a, b)$ and the rate of change is $|\nabla f(a, b)|$
- ② f has max rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$ and the rate of change is $|\nabla f(a, b)|$
- ③ f has no change at (a, b) in any direction that is orthogonal to $\nabla f(a, b)$

Thm 13.12. Assume $\nabla f(a, b) \neq 0$. The line tangent to the level curve at (a, b) is orthogonal to $\nabla f(a, b)$

★ Example:

$$f(x, y) = 4 - x^2 - 2y^2 \quad P(1, 1)$$

① Find $\nabla f(1, 1)$

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \langle -2x, -4y \rangle$$

$$\nabla f(1, 1) = \langle -2, -4 \rangle$$

Q27 Find directional derivative at P in the direction of $\vec{u} = \langle -1, 2 \rangle$

Answer: $D_{\vec{u}}f(1,1) = \nabla f(1,1) \cdot \frac{\vec{u}}{|\vec{u}|}$
 $= \langle -2, -4 \rangle \cdot \frac{\langle -1, 2 \rangle}{\sqrt{5}}$
 $= -\frac{6}{\sqrt{5}}$

Q37 What's the unit vector in the direction of steepest ascent at P? What's the rate of increase?

Answer: direction of steepest ascent = $\frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \frac{\langle -2, -4 \rangle}{\sqrt{4+16}} = \frac{1}{\sqrt{5}} \langle -1, -2 \rangle$

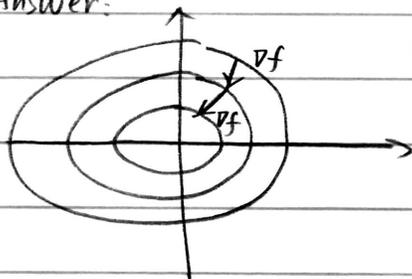
Q47 Can you find a direction that f has no change at P?

Answer: In any direction $\perp \nabla f(1,1)$, f has no change
 $\langle 4, -2 \rangle$ or $\langle -4, 2 \rangle$ are two examples.

Recall $\langle a, b \rangle \cdot \langle -b, a \rangle = 0$
 $\langle a, b \rangle \cdot \langle b, -a \rangle = 0$

Q57 Let C' be the path of steepest descent on the surface beginning at P and C is the projection of C' on xy-plane. Find the equation of C

Answer:



∇f provides the tangent direction of curve C

$$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{-4y}{-2x} = \frac{2y}{x}$$

$$\frac{dy}{y} = \frac{2dx}{x}$$

$$\ln y = 2 \ln x + C$$

$$y = e^C x^2$$

Since $P(1,1)$ is on the curve, plug in $x=1$ and $y=1$ to solve for e^C

$$\Rightarrow y = x^2$$

Q67 The parametric equation for C' on the surface

Answer $\vec{r}(t) = \langle t, t^2, 4 - t^2 - 4 + 4 \rangle$ #

Def: Tangent plane at (a,b,c) of a surface $F(x,y,z)=0$ has the normal vector

$$\vec{n} = \nabla F(a,b,c) = \langle F_x(a,b,c), F_y(a,b,c), F_z(a,b,c) \rangle$$

Special case: $F(x,y,z)=0$ has the form $z=f(x,y)$. Then $F(x,y,z)=f(x,y)-z=0$

$$\vec{n} = \nabla F = \langle f_x, f_y, -1 \rangle$$

Equation of tangent plane $F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$

• Linear approximation $L(x,y)$ at (a,b) of a surface $z=f(x,y)$ is the tangent plane

$$L(x,y) = z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

We can use $L(x,y)$ to estimate $z=f(x,y)$ at (x_0,y_0) near (x,y)

$$f(x_0,y_0) \approx L(x_0,y_0) = f_x(a,b)(x_0-a) + f_y(a,b)(y_0-b) + f(a,b)$$

Examples HW 8 # 13 # 15 # 16

#13 $x^2 + y^2 - z^2 - 94 = 0$ (7,9,16)

Answer: $\vec{n} = \nabla F(7,9,16) = \langle 2x, 2y, -2z \rangle \Big|_{(7,9,16)}$
 $= \langle 14, 18, -32 \rangle$

$$14(x-7) + 18(y-9) - 32(z-16) = 0$$

#15 $z = \ln(1+xy)$ (1,4,ln5)

$$f_x = \frac{y}{1+xy} \quad f_y = \frac{x}{1+xy}$$

Answer $\vec{n} = \langle f_x(1,4), f_y(1,4), -1 \rangle$
 $= \langle \frac{4}{5}, \frac{1}{5}, -1 \rangle$

$$\frac{4}{5}(x-1) + \frac{1}{5}(y-4) - (z - \ln 5) = 0$$

#16 $f(x,y) = 3x - 4y + 6xy$ P(2,4)

① $L(x,y) = f_x(2,4)(x-2) + f_y(2,4)(y-4) + f(2,4)$

$$f_x = 3 - 6y$$

$$L(x,y) = -2(x-2) + 8(y-4) + 38$$

$$f_y = -4 + 6x$$

② $L(2.1, 4.01) = -2(2.1-2) + 8(4.01-4) + 38$

$$6 - 26 + 12 \times 4$$

$$= -2 \times 0.1 + 8 \times 0.01 + 38$$

$$-10 + 48$$